NOTES ON AUCTIONS AND BIDDING

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1. Auctions and Bidding - Introduction

Auction: Explicit set of rules determining allocation on the basis of bids

Orgnization of “thin” markets with market power on one side: One side of the market has “power” to announce a set of rules and commit itself to these rules.

Examples

Buying/Procurement: bridges, highways, hospitals, fighter planes, trains, consulting services, etc., etc.
Selling/Auctions: real estate, stocks, firms, wine, art, fish, cars, land, slaves, licenses (oil drilling, airwaves), etc., etc.
Buying and selling on the Internet - online auctions: B2B, B2C, B2G

Examples from ancient Rome

(A) Provinces, or rather, the right to collect taxes (tax farming)
(B) The Empire itself: Sold by Praetorian Guard to Didius Julianus in AD 193 (the price is known, see Gibbon, *The Decline and Fall of the Roman Empire*, ch. 5)

More generally: Auction-like institutions such as
- Auctions
- Procurement auctions
- Double-auctions
- Structured negotiation
- Organized exchanges

are suitable for eliciting private information.

Generally, agents have incentives to conceal their preferences, costs, etc.

Used-car example

One seller and several potential buyers. Seller does not know the willingness to pay of the buyers, and the buyers do not know each others willingness to pay. Ascending bid auction results in the bidder with the highest valuation paying approximately the valuation of the bidder with the second-highest valuation. [We show this below!]

Compare this with the possible alternative where the seller simply posts a price, and the car is sold on a first-come, first-served basis (provided someone is willing to pay the posted price).

1.1. Standard auction types

For selling (for buying is the same with the necessary “change of signs”)

- Open, ascending bid auction (OAB)

English or Japanese - cars, art, wine, fish (DK), Internet Auctions (*Yahoo!*, *eBay* (maybe))

Explain the details and the theoretical ideal

Used-car example

- Open, descending bid auction (ODB)

Dutch - fish (several countries), cut flowers

Explain the details

- First-price, sealed-bid auction (FPSB or, simply, FP)

High-bid auction - winner pays his bid.

Public procurement - roadwork, heating pipes, bridges, building contracts, consulting contracts

Explain the details

- Second-price, sealed-bid auction (SPSB or, simply, SP))

Second-bid or Vickrey auction - winner pays the highest losing bid (in the general formulation) - many variations, but key feature is that a winning bidder does not pay his own bid

Explain the details

eBay auctions in light of this

Danish UMTS auction in light of this: ("Close") Winners pay the fourth-highest/lowest winning bid rather than the fifth highest/highest losing bid.

General comment on Vickrey-Clarke-Groves Mechanisms

1.2. Variations on standard types

- reserve price (open or secret, phantom bids (e.g., livestock, thorough-breds))
- entry/participation fee
- minimum increase/bid increment
- non-zero payments by non-winners
- activity rules
- fixed termination

- off-line - online (*eBay*'s handling - a hidden reported maximum/proxy bidding - incentives?)

All-pay auctions
Interpret
Economics/auctions
R&D race
War of attrition

Anglo-Dutch format: Explain!

1.3. Main question

Suppose a single seller has *one* unit of an indivisible good.

> What is an optimal selling institution for this good when the seller does not know the valuations of the potential buyers?

Clearly, it is not generally optimal to simply post a fixed price! A moment's reflection should convince you of this.

Optimality

From the point of view of the private seller?
Ex. A rare coin: Presumably, the seller wants to maximize revenue (in expected terms? corrected for risk?)

From the point of view of society in general?
Ex. UMTS licenses: Presumably, the initial owner of the radio frequencies (society) wants to reallocate rights (to use) to greatest possible benefit of society as a whole (again, there is uncertainty and some possible risk considerations that have to be factored in).

In any case, the seller wants to design an institution to maximize the expected utility/payoff!

This implies that auction design is an exercise in Mechanism Design.
Mechanism Design forms a whole branch of the economics literature.

Often: We simply want to compare different institutions/auction designs in terms of the expected revenues of the seller.

Ultimately: We want to design an optimal auction/mechanism.

1.4. Buyer valuations

Throughout, buyers have private information about their valuations.

Mostly, we shall assume that a particular buyer only wants to buy one unit
Only one unit for sale
Buyer only allowed to buy one (e.g. UMTS)

Hence, the valuation of the buyer is well-defined (maximum willingness to pay).

Cases: Single Unit

I. Independent Private Values (IPV) Model

- Each bidder in an auction *knows* precisely how he values the object (and this is independent of how others value it)
- A bidder, however, does not know the values of other bidders
- A bidder considers the values of others as independent, random draws from distributions which are common knowledge (all bidders typically draw from *identical* distributions - IIPV Model)

II. Common Values (CV) Model

- The object has a "true" value common to all, and the bidders have different and private information about this value

III. General/"Correlated"/"Affiliated" Values Model

- Combine elements of private and common values - "Correlation"/Affiliation

In the following, we focus mostly on I.

The general, affiliated values case.
Learning from rivals
Experts vs. non-experts
Timing of bids (experts go late)
Preferences over transparency vs. opaqueness
Non-experts might prefer transparent auction set-up, while experts might not
The auction designer has to take this into account when setting up the auction format

The *eBay* deal engine
- Fixed termination

- Highest bid not displayed (only second-highest + bid increment)
- This allows “secret” last minute bids by the experts

→ Experts are happy with this format

Contrast with *Yahoo!* format

- No fixed termination
- 10 minutes to respond to high bid
- More transparent

→ Experts are less happy with this format (their margins are squeezed)

This might explain the success of *eBay* compared to auctions at *Yahoo!*

1.5. Further issues

Auction theory touches upon a number of further issues, which we briefly discuss in this note.

1.5.1. Risk attitudes

Bidder risk aversion
Seller risk aversion

See below.

1.5.2. Asymmetric bidders

See below.

1.5.3. Multi-unit auctions

Supply is fixed

Lots: Could collect in one lot and auction it off in one go - But not typically optimal!
Instead:
Ask for generalized bids (price and quantity) - individual demand functions
Generate a market demand function and fill orders from the top
Looks like a generalized ascending bid auction

What about prices?

(1) *Pacman* price discrimination

(i) Winning bidders pay their bid for each unit won
Like a generalization of the first-price auction for a single unit
Hence, price coincides with bid for each unit, and there is price discrimination
(ii) Winning bidders pay the bid immediately below for each unit

Generalization of the second-price auction for a single unit
Hence, winning bids do not coincide with price, but there is still price discrimination

(2) One price (price is defined by crossing of market demand with fixed supply)
Price equals lowest winning bid, and there is no price discrimination!
Winning bids do not coincide with price, but do of course decide whether a bidder is a winner
This has a Vickrey flavor!
Used by *eBay*

Partial fills??
Activity rules??

Simultaneous vs. sequential auctions
Sequential auction: First one lot sold, then the next, and so on
Aggregation - e.g. radiofrequencies, wine
Complementarity
Combinatorial bids

Comments on UMTS
Fixed supply of radiofrequencies
Number of winners fixed or endogenous? (cp. UK and Germany)
Asymmetric bidders (incumbents (2G) and potential entrants)
Restrictions on bidders
- exclusion (discussion in UK case and recent OECD report)
- reserved license
Format:
- open ascending bid
- sealed bid
- Anglo-Dutch

For more, see below.

1.5.4. Two-sided, multi-unit auctions

Supply is not fixed
Double auctions - Organized exchanges
Commodities and securities exchanges
Potential for B2B??
Electricity

Format
"Neutral" market maker (the exchange)

Ask for generalized bids (demands) and asks (supplies)
Aggregate bids into market demand and asks into market supply
Clear market by equating demand and supply
Buyers who bid more than or equal to the market clearing price are served
Sellers who ask less than or equal to the market clearing price are asked to supply

The price??

(1) One price
Price equals the market clearing price defined above
Market maker is financed by a common participation fee from active traders

(2) Price discrimination
(Winning) Buyers pay their bids for each unit
(Winning) Sellers receive their asks for each unit
Or some variation on this
Market maker collects the surplus

For more, see below.

1.5.5. "Gaming" the mechanism

(1) Shill bidding
Bidding by seller in e.g. *eBay* to raise price above the "legitimate" second-highest price or market clearing price
Particularly relevant when multiple units are on offer
Compare to reserve price - secret or open
Compare also to the standard monopolist who wants to raise price above his unit costs (which may be zero) - he reduces supply

(2) Offers outside the online auction
To avoid auction fee a seller of e.g. baseball tickets may lurk in the wings and contact bidders directly (if their identity is revealed)

(3) Leaking of "proxy" bids - revise them downwards, or update as the auction progresses

(4) Bidder collusion
Bidding Rings
Central heating pipes in DK??
Several recent DK cartel cases??
Ohio school milk??
Long Island road work??
Dutch UMTS??

(5) Bidder signaling and communication

US PCS auctions??
German 2G auction (for additional spectrum)??
German UMTS??

(6) Default and retraction of bids
Australian TV licenses??
Early experiences from online auctions (rare coins at *eBay*)

(7) Colluding auctioneers
Sotheby's and *Christie*'s auctions??

For more, see below.

2. IIPV Model - Single-Unit Case

The identically (I) distributed, independent (I) private (P) values (V) model (IIPV-benchmark).

Assumptions

A1: Bidders are risk neutral
A2: Values are private and independently distributed
A3: Bidders are symmetric (ex ante identical)
A4: Payments depend on bids alone (non-discrimination)
[A5: Seller is risk neutral]
[A6: There is a single, indivisible unit for sale]

Postulate some trading institution (a mechanism) and look for BNE bidding strategies.

BNE - Bayesian Nash Equilibrium

In equilibrium, bidders submit optimal bids as functions of their valuations (types) and correct expectations about the bidding strategies of the other bidders.

2.1. Preliminaries

1st Result. Formally, i.e. from a game theory perspective, the Dutch and the FPSB auctions are equivalent. Thus, bidding strategies and payoffs should be identical.

"Proof". In both formats bidders have to make bids without learning anything from the choices of others. The two formats have the same reduced normal form (→ Strategic Equivalence). Heuristically, whether I sit in an auction room waiting for the price to fall to my strike price (bid) or whether I leave my bid (strike price) in a sealed envelope (an go on a picnic) is inconsequential. ■

So, we can consider Dutch and FPSB auctions as one, and we compare OAB, SPSB and FPSB auctions in the following.

Formalization

- n - bidders (potentiel buyers)
- v_i - bidder i's valuation
- v_i is drawn independently from *common* probability distribution $F_i = F$, $i = 1, 2, 3, ..., n$, with continuous density f $(= F')$ on $[\underline{v}, \overline{v}]$. This captures *symmetry.*
- Suppose we have n draws from F such that $V_{[1]} \geq V_{[2]} \geq ...... \geq V_{[n]}$. This is just a *labelling* of the highest valuation drawn, the second-highest valuation drawn, and so on.

OAB Auction

In an OAB auction it is a *weakly dominant* strategy for bidder i with valuation v_i to remain in the bidding until the price, p, exceeds v_i. $\Rightarrow$

$$\text{Seller Revenue} = V_{[2]} \text{ (or } V_{[2]} + \varepsilon)$$

$$\begin{array}{rcll} \text{Buyer Payoffs} & = & V_{[1]} - V_{[2]} & \text{to winner} \\ & & 0 & \text{to all losers} \end{array}$$

Hence,

$$\begin{array}{rl} & \text{Expected payoff to participating bidder} \\ = & \text{Expected value of } V_{[1]} - V_{[2]} \\ = & \{\text{probability of winning}\} \times \{\text{expected surplus when winning}\} \end{array}$$

SPSB Auction

In an SPSB auction it is a *weakly dominant* strategy to bid one's value v_i. Why?

- Bidding $b_i = v_i$ gives $v_i - \max_{j \neq i}\{b_j\}$ if $v_i > \max_{j \neq i}\{b_j\}$ and 0 otherwise (tie-breaking?).
- Bidding $b_i > v_i$ is *suboptimal*, since there is a strictly positive probability that $b_i > \max_{j \neq i}\{b_j\} > v_i$ in which case $v_i - \max_{j \neq i}\{b_j\} < 0$ and i is *winner.*
- Bidding $b_i < v_i$ is *suboptimal*, since there is a strictly positive probability that $v_i > \max_{j \neq i}\{b_j\} > b_i$ in which case $v_i - \max_{j \neq i}\{b_j\} > 0$ and i is *not winner.*

2nd Result. The OAB and SPSB auctions are equivalent, in the sense that the optimal bidding strategies are essentially the same (in the IIPV case). Hence, they give rise to the same expected payments and revenues.

"Proof". Almost trivial. In the OAB auction, the bidder with the 2nd highest value drops out at $p = V_{[2]}$, and the highest valuation bidder wins and pays $V_{[2]}$. The expected revenue is the expectation of $V_{[2]}$. In the SPSB auction, $b_i = v_i$, $\forall i$, and the expected revenue is the expectation of $V_{[2]}$. The highest valuation bidder bids $V_{[1]}$ and pays $V_{[2]}$ to obtain the object. ■

Remark: In OAB and SPSB auctions, the payments of the winner can be said to be independent of his own bid.

Summing up so far (1st Result and 2nd Result)

- Dutch and FPSB auctions are strategically equivalent ($\rightarrow$ same strategies/same equilibrium payoffs).
- OAB and SPSB auctions give rise to the same expected revenue and the same expected *rent* to winning bidder $E\{V_{[1]} - V_{[2]}\}$.
- So, what remains is a comparison of Dutch and FPSB auctions on the one hand and OAB and SPSB auctions on the other.

But first, a few comments on Order Statistics.

2.2. Order statistics

[Based on Krishna, 2002, Appendix C.] Consider n independent random draws from the same distribution on $[\underline{v}, \overline{v}]$ represented by F. As above, we order the valuations drawn such that $V_{[1]} \geq V_{[2]} \geq \ldots\ldots \geq V_{[n]}$. $V_{[k]}$ is referred to as the kth-order statistic. We shall occasionally refer to this as $V_{[k,n]}$ to capture that it is the kth-order statistic in a sample of n. Below, it will be useful to know the distributions of $V_{[1]}$ and $V_{[2]}$ and the relationship between their distributions.

Distribution of the first-order statistic $V_{[1]}$

Let $F_{1,n}(v)$ denote the distribution function of $V_{[1]}$ (the highest of n draws from $F(v)$).

Now, $F_{1,n}(v)$ is just the probability that all the n draws are less than or equal to v. Hence,

$$F_{1,n}(v) = (F(v))^n = F^n(v) \tag{2.1}$$

It follows that the density, $f_{1,n}(v)$, of $V_{[1]}$ is

$$f_{1,n}(v) = \frac{dF_{1,n}(v)}{dv} = \frac{dF^n(v)}{dv} = nf(v)F^{n-1}(v) \tag{2.2}$$

Combinatorially, this may be thought of as the n ways that one value could be v, while the rest are lower.

The expected value of $V_{[1]}$ is

$$E(V_{[1]}) = \int_{\underline{v}}^{\overline{v}} vf_{1,n}(v)dv = \int_{\underline{v}}^{\overline{v}} nvf(v)F^{n-1}(v)dv \tag{2.3}$$

Distribution of the second-order statistic $V_{[2]}$

Let $F_{2,n}(v)$ denote the distribution function of $V_{[2]}$ (the second-highest of n draws from $F(v)$).

Now, $F_{2,n}(v)$ is the probability that $V_{[2]}$ is less than or equal to v. This, however, is the union of two disjoint events: a) all n values are less than or equal to v, and b) $n-1$ values are less than or equal to v and one value is greater than v. Hence,

$$F_{2,n}(v) = F^n(v) + nF^{n-1}(v)(1 - F(v)) \tag{2.4}$$

We can rewrite this as

$$F_{2,n}(v) = F^{n-1}(v)(n - (n-1)F(v)) \tag{2.5}$$

Therefore, the density, $f_{2,n}(v)$, of $V_{[2]}$ is

$$f_{2,n}(v) = \frac{dF_{2,n}(v)}{dv} = \frac{d(F^{n-1}(v)(n - (n-1)F(v)))}{dv}$$

Thus,

$$f_{2,n}(v) = (n-1)f(v)F^{n-2}(v)(n - (n-1)F(v)) - F^{n-1}(v)(n-1)f(v)$$

which reduces to

$$f_{2,n}(v) = n(n-1)f(v)(1 - F(v))F^{n-2}(v) \tag{2.6}$$

If we write this density as $nf(v) \times (n-1)(1 - F(v)) \times F^{n-2}(v)$, it may be thought of as the n ways that one value could be v combined with the $n-1$ ways that one could be higher and the remaining $n-2$ lower.

The expected value of $V_{[2]}$ is

$$\begin{aligned} E(V_{[2]}) &= \int_{\underline{v}}^{\overline{v}} vf_{2,n}(v)dv \\ &= \int_{\underline{v}}^{\overline{v}} n(n-1)vf(v)(1 - F(v))F^{n-2}(v)dv \end{aligned} \tag{2.7}$$

Relationships

We can rewrite $F_{2,n}(v)$ as

$$\begin{aligned} F_{2,n}(v) &= nF^{n-1}(v) - (n-1)F^n(v) \\ &= nF_{1,n-1}(v) - (n-1)F_{1,n}(v) \end{aligned}$$

from which it follows immediately that

$$f_{2,n}(v) = nf_{1,n-1}(v) - (n-1)f_{1,n}(v)$$

and

$$E(V_{[2,n]}) = nE(V_{[1,n-1]}) - (n-1)E(V_{[1,n]})$$

We also note that

$$\begin{aligned} f_{2,n}(v) &= n(n-1)f(v)(1-F(v))F^{n-2}(v) \\ &= n(1-F(v))(n-1)F^{n-2}(v)f(v) \end{aligned}$$

Hence,

$$f_{2,n}(v) = n(1-F(v))f_{1,n-1}(v)$$

Joint density of $V_{[1,n]}$ *and* $V_{[2,n]}$

Let $y_1 \geq y_2 \geq ...... \geq y_n$. Then the joint density of y_1 and y_2 is given by

$$f_{1,2,n}(y_1, y_2) = n(n-1)f(y_1)f(y_2)F^{n-2}(y_2)$$

Density of $V_{[2,n]}$ conditional on $V_{[1,n]} = y$ is (for $y > z$)

$$\begin{aligned} f_{2,n}(z \quad | \quad V_{[1,n]} = y) = \frac{f_{1,2,n}(y,z)}{f_{1,n}(y)} \\ &= \frac{n(n-1)f(y)f(z)F^{n-2}(z)}{nf(y)F^{n-1}(y)} \\ &= \frac{(n-1)f(z)F^{n-2}(z)}{F^{n-1}(y)} \end{aligned}$$

Also, the density of $V_{[1,n-1]}$ conditional on $V_{[1,n-1]} < y$ is given by

$$\begin{aligned} f_{1,n-1}(z \quad | \quad V_{[1,n-1]} < y) = \frac{f_{1,n-1}(z)}{F_{1,n-1}(y)} \\ &= \frac{(n-1)f(z)F^{n-2}(z)}{F^{n-1}(y)} \end{aligned}$$

and we conclude that

$$f_{2,n}(\cdot \mid V_{[1,n]} = y) = f_{1,n-1}(\cdot \mid V_{[1,n-1]} < y)$$

In words: The distribution of the *second-order* statistic in a sample of n *conditioned* on the *first-order* statistic being y is *equivalent* to the distribution of the *first-order* statistic in a sample of $n-1$ *conditioned* on it being less than y. This will prove useful later.

2.3. Bidding strategies and seller revenues

a) *OAB/SPSB Auctions*

We can write the payoff of bidder i as

$$\begin{aligned} u_i &= v_i - \max_{j \neq i} b_j \quad if \quad b_i > \max_{j \neq i} b_j \\ &= 0 \quad if \quad b_i < \max_{j \neq i} b_j \end{aligned}$$

If several bidders have the high bid, lots are drawn. We have already established the following result.

Proposition 1. *In a second-price, sealed-bid auction, it is a weakly dominant strategy for each bidder to bid according to*

$$b = b_{SP}(v) = v \tag{2.8}$$

b) *Dutch/FPSB Auctions*

We can write the payoffs of bidder i as

$$\begin{aligned} u_i &= v_i - b_i \quad if \quad b_i > \max_{j \neq i} b_j \\ &= 0 \quad if \quad b_i < \max_{j \neq i} b_j \end{aligned}$$

If several bidders have the high bid, lots are drawn.

No BNE in dominant strategies in this case. So, what is a BNE?

Suppose bidder i anticipates that all other bidders, $j \neq i$, bid according to the same bidding function $b(\cdot)$. Hence, $b_j = b(v_j)$, $\forall j \neq i$.

Assumption: $b(\cdot)$ is monotone increasing

What is the optimal bid of bidder i given he has valuation v_i? Suppose he bids b_i. Then, i's payoff can be written as

$$u_i = (v_i - b_i) \times \{\text{probability of winning}\}$$

where

$$\begin{aligned} & \{\text{probability of winning}\} \\ = & \{\text{probability that everybody else bid less than } b_i \text{ according to } b(v_j)\} \end{aligned}$$

Now

$$\begin{aligned} & pr(j \text{ bids less than } b_i) \\ = & pr(b(v_j) < b_i) \\ = & pr(v_j < b^{-1}(b_i)) \\ = & F(b^{-1}(b_i)) \end{aligned}$$

It follows immediately that

$$\begin{aligned} & pr(\forall j \text{ bid less than } b_i) \\ = & [F(b^{-1}(b_i))]^{n-1} \end{aligned}$$

Notation: Let $F^m(x) = [F(x)]^m$

$$\Rightarrow u_i = (v_i - b_i)F^{n-1}(b^{-1}(b_i))$$

Now, let $b_i = b_i(v_i)$, where $b_i(\cdot)$ is some function of v_i (not necessarily the same as $b(\cdot)$). Then, if bidder i pretended to have valuation r, he would bid $b_i(r)$. So, we can define the payoffs to bidder i with actual valuation v_i when he pretends to have valuation r as follows

$$u_i = (v_i - b_i(r))F^{n-1}(b^{-1}(b_i(r)))$$

We look for a BNE in symmetric, strictly increasing strategies, i.e. $b_i(r) = b(r)$.

$$\begin{aligned} \Rightarrow \quad & u_i = (v_i - b(r))F^{n-1}(b^{-1}(b(r))) \\ = \quad & (v_i - b(r))F^{n-1}(r) = (v_i - b(r))F_{1,n-1}(r) \end{aligned}$$

where $F^{n-1}(r)$ is the probability of winning with a bid of $b(r)$.

Bidding strategies in Dutch/FPSB auctions? We can state the following result.

Proposition 2. *Symmetric equilibrium strategies of a first-price, sealed-bid auction are given by*

$$b_{FP}(v) = \frac{\int_{\underline{v}}^{v} x(n-1)f(x)F^{n-2}(x)dx}{F^{n-1}(v)} \tag{2.9}$$

$$b_{FP}(v) = v - \frac{\int_{\underline{v}}^{v} F^{n-1}(x)dx}{F^{n-1}(v)} \tag{2.10}$$

$$b_{FP}(v) = \frac{\int_{\underline{v}}^{v} x f_{1,n-1}(x)dx}{F_{1,n-1}(v)} \tag{2.11}$$

In this proposition, we have written the bidding strategy in three different ways. This will be useful below. But first, we approach the proof of the proposition in three different ways.

i) *Jehle & Reny's "direct mechanism approach"*

Suppose, hypothetically, that bidder i cannot attend the auction and intructs an agent to bid according to $b(\cdot)$ on his behalf. Bidder i tells the agent that he has valuation r. In a symmetric equilibrium it must be optimal for bidder i to set $r = v_i$, that is, to tell the agent the truth. In other words

$$r = v_i \text{ must solve } \max_r \{u_i(r, v_i)\}$$

Now,

$$\begin{aligned} \tfrac{du_i(r,v_i)}{dr} &= \tfrac{d(v_i - b(r))F^{n-1}(r)}{dr} \\ &= (v_i - b(r))(n-1)F^{n-2}(r)f(r) - b'(r)F^{n-1}(r) \end{aligned}$$

Setting the derivative to zero at $r = v_i$ gives

$$(v_i - b(v_i))(n-1)F^{n-2}(v_i)f(v_i) = b'(v_i)F^{n-1}(v_i)$$

which can be written as

$$(n-1)v_i f(v_i)F^{n-2}(v_i) = (n-1)b(v_i)f(v_i)F^{n-2}(v_i) + b'(v_i)F^{n-1}(v_i)$$

But

$$\frac{db(v_i)F^{n-1}(v_i)}{dv_i} = (n-1)b(v_i)f(v_i)F^{n-2}(v_i) + b'(v_i)F^{n-1}(v_i)$$

and we conclude that

$$\frac{db(v_i)F^{n-1}(v_i)}{dv_i} = (n-1)v_i f(v_i)F^{n-2}(v_i)$$

This must hold for all $v_i \in [\underline{v}, \overline{v}]$. Hence,

$$b(v_i)F^{n-1}(v_i) = (n-1)\int_{\underline{v}}^{v_i} x f(x)F^{n-2}(x)dx + const$$

For $v_i = \underline{v}$, we have

$$\begin{aligned} b(\underline{v})F^{n-1}(\underline{v}) &= (n-1)\int_{\underline{v}}^{\underline{v}} x f(x)F^{n-2}(x)dx + const \\ &\Rightarrow 0 = 0 + const \\ &\Rightarrow const = 0 \end{aligned}$$

Therefore, we have

$$b(v_i)F^{n-1}(v_i) = (n-1)\int_{\underline{v}}^{v_i} xf(x)F^{n-2}(x)dx$$

and we have

$$b_{FP}(v_i) = \frac{\int_{\underline{v}}^{v_i} x(n-1)f(x)F^{n-2}(x)dx}{F^{n-1}(v_i)} \tag{2.12}$$

This gives the optimal bid in a Dutch/FPSB auction as a function of v_i. Note that this is (2.9) in the proposition.

We can rewrite this as

$$\begin{aligned} b_{FP}(v_i) &= v_i - v_i + \tfrac{1}{F^{n-1}(v_i)} \textstyle\int_{\underline{v}}^{v_i} x(n-1)f(x)F^{n-2}(x)dx \\ &= v_i - \tfrac{1}{F^{n-1}(v_i)}[v_iF^{n-1}(v_i) - \textstyle\int_{\underline{v}}^{v_i} x(n-1)f(x)F^{n-2}(x)dx] \\ &= v_i - \tfrac{1}{F^{n-1}(v_i)}[[v_iF^{n-1}(v_i) - \underline{v}F^{n-1}(\underline{v})] - \textstyle\int_{\underline{v}}^{v_i} x(n-1)f(x)F^{n-2}(x)dx] \end{aligned}$$

$$b_{FP}(v_i) = v_i - \frac{1}{F^{n-1}(v_i)}[[xF^{n-1}(x)]_{\underline{v}}^{v_i} - \int_{\underline{v}}^{v_i} x(n-1)f(x)F^{n-2}(x)dx] \tag{2.13}$$

We note that

$$\frac{dF^{n-1}(x)}{dx} = (n-1)F^{n-2}(x)f(x)$$

and (by partial integration)

$$\int_{\underline{v}}^{v_i} x(n-1)f(x)F^{n-2}(x)dx = [xF^{n-1}(x)]_{\underline{v}}^{v_i} - \int_{\underline{v}}^{v_i} 1 \cdot F^{n-1}(x)dx$$

hence,

$$[xF^{n-1}(x)]_{\underline{v}}^{v_i} - \int_{\underline{v}}^{v_i} x(n-1)f(x)F^{n-2}(x)dx = \int_{\underline{v}}^{v_i} F^{n-1}(x)dx$$

We substitute this into (2.13) to obtain the bidding function in the more convenient form

$$b_{FP}(v_i) = v_i - \frac{\int_{\underline{v}}^{v_i} F^{n-1}(x)dx}{F^{n-1}(v_i)} \tag{2.14}$$

which is just (2.10) in the proposition. From this it is immediate that the optimal bid in the FPSB auction is a *mark-down* on the valuation v_i.

ii) *Alternative derivation ("brute force" maximization)*

Recall the general expression for bidder i's payoffs

$$u_i = (v_i - b_i)F^{n-1}(b^{-1}(b_i))$$

To find the optimal bid given v_i, we maximize u_i with respect to b_i. At optimum we must have

$$\frac{\partial u_i}{\partial b_i} = 0$$

Then, by the Envelope Theorem,

$$\frac{du_i}{dv_i} = \frac{\partial u_i}{\partial v_i} + \frac{\partial u_i}{\partial b_i} \cdot \frac{db_i}{dv_i} = \frac{\partial u_i}{\partial v_i} + 0 \cdot \frac{db_i}{dv_i} = \frac{\partial u_i}{\partial v_i} = F^{n-1}(b^{-1}(b_i))$$

In *symmetric* equilibrium, bidder i's assessment that $b_j = b(v_j)$, $\forall j \neq i$, is correct, and $b_i = b(v_i)$. Thus, $v_i = b^{-1}(b_i))$, and we have

$$\frac{du_i}{dv_i} = F^{n-1}(b^{-1}(b_i)) = F^{n-1}(v_i)$$

This is a differential equation, and we obtain

$$u_i = u_i(\underline{v}) + \int_{\underline{v}}^{v_i} \frac{du_i(t)}{dt} dt$$

Where $u_i(\underline{v}) = 0$ is the payoff to the lowest valuation bidder (with no chance of winning)

$$\Rightarrow u_i = \int_{\underline{v}}^{v_i} \frac{du_i(x)}{dx} dx$$

So, we have *two* expressions for u_i,

$$u_i = (v_i - b(v_i))F^{n-1}(b^{-1}(b_i)) = (v_i - b(v_i))F^{n-1}(v_i) \tag{2.15}$$

and

$$u_i = \int_{\underline{v}}^{v_i} \frac{du_i(x)}{dx} dx = \int_{\underline{v}}^{v_i} F^{n-1}(x)dx \tag{2.16}$$

We equate the right-hand-sides of (2.15) and (2.16) to get

$$(v_i - b(v_i))F^{n-1}(v_i) = \int_{\underline{v}}^{v_i} F^{n-1}(x)dx$$

$$\Rightarrow b_{FP}(v_i) = v_i - \frac{\int_{\underline{v}}^{v_i} F^{n-1}(x)dx}{F^{n-1}(v_i)}$$

Which is just (2.10) in the proposition.

iii) *Alternative derivation (using order statistics notation)*

We look for a symmetric equilibrium with $b_i = b_{FP}(v_i)$ for all i. Bidder i assumes that the other bidders, $j \neq i$, bid according to $b_{FP}(v_j)$. Assume that bidder i has valuation v and bids b. We want to determine the optimal b. Note that $b_{FP}(0) = 0$.

Bidder i wins whenever $\max_{j \neq i} b_{FP}(v_j) < b$. Since $b_{FP}(\cdot)$ is strictly increasing (checked below), this is equivalent to $\max_{j \neq i} v_j < b_{FP}^{-1}(b)$. This event has probability $F^{n-1}(b_{FP}^{-1}(b))$, and we can write bidder i's expected payoffs as a function of his bid

$$\begin{aligned} u(b; v) &= pr\{winning\} \times (v - b) \\ &= F^{n-1}(b_{FP}^{-1}(b))(v - b) \\ &= F_{1,n-1}(b_{FP}^{-1}(b))(v - b) \end{aligned}$$

Take derivative and set to 0

$$\begin{aligned} \frac{du(b; v)}{db} &= \frac{dF_{1,n-1}(b_{FP}^{-1}(b))}{dx} \frac{db_{FP}^{-1}(b)}{db}(v - b) \\ &\quad - F_{1,n-1}(b_{FP}^{-1}(b)) \\ &= 0 \end{aligned}$$

$$\frac{f_{1,n-1}(b_{FP}^{-1}(b))}{b'_{FP}(b_{FP}^{-1}(b))}(v - b) - F_{1,n-1}(b_{FP}^{-1}(b)) = 0 \qquad (2.17)$$

In a symmetric equilibrium, we must have $b = b_{FP}(v)$. This can be written as $b_{FP}^{-1}(b) = v$. Substituting into (2.17) we get

$$\frac{f_{1,n-1}(v)}{b'_{FP}(v)}(v - b_{FP}(v)) - F_{1,n-1}(v) = 0$$

which we can write as

$$b'_{FP}(v)F_{1,n-1}(v) + b_{FP}(v)f_{1,n-1}(v) = vf_{1,n-1}(v)$$

or

$$\frac{d(b_{FP}(v)F_{1,n-1}(v))}{dv} = vf_{1,n-1}(v)$$

This, in turn, implies that

$$\begin{aligned} b_{FP}(v)F_{1,n-1}(v) &= b_{FP}(\underline{v})F_{1,n-1}(\underline{v}) \\ &\quad + \int_{\underline{v}}^{v} xf_{1,n-1}(x)dx \end{aligned}$$

Since the first term on the right-hand-side is 0 ($F_{1,n-1}(\underline{v}) = 0$), we can write

$$b_{FP}(v) = \frac{\int_{\underline{v}}^{v} xf_{1,n-1}(x)dx}{F_{1,n-1}(v)}$$

as claimed in (2.11) of the proposition.

Now, (2.17) is merely a necessary condition, but we claim that it is, indeed, optimal for bidder i to bid $b_{FP}(v)$, given that other bidders follow $b_{FP}(\cdot)$.

Why? First recall that $b_{FP}(\cdot)$ is strictly increasing, which implies that the highest bidder wins if everybody bids according to $b_{FP}(\cdot)$. Now, suppose that bidder i bids $b \neq b_{FP}(v)$, and let r be the (hypothetical) valuation for which $b_{FP}(r) = b$. That is, $r = b_{FP}^{-1}(b)$. Then, we can write bidder i's payoff from bidding $b = b_{FP}(r)$ when his actual valuation is v as

$$\begin{aligned} u(b;v) &= u(b_{FP}(r);v) = F_{1,n-1}(r)(v - b_{FP}(r)) \\ &= F_{1,n-1}(r)v - \int_{\underline{v}}^{r} x f_{1,n-1}(x)dx \\ &= F_{1,n-1}(r)v \\ &\quad -[[xF_{1,n-1}(x)]_{\underline{v}}^{r} - \int_{\underline{v}}^{r} F_{1,n-1}(x)dx] \\ &= F_{1,n-1}(r)(v-r) + \int_{\underline{v}}^{r} F_{1,n-1}(x)dx \end{aligned}$$

Hence,

$$\begin{aligned} &u(b_{FP}(v);v) - u(b_{FP}(r);v) \\ =\ & F_{1,n-1}(v)(v-v) + \int_{\underline{v}}^{v} F_{1,n-1}(x)dx \\ & -F_{1,n-1}(r)(v-r) - \int_{\underline{v}}^{r} F_{1,n-1}(x)dx \\ =\ & F_{1,n-1}(r)(r-v) - \int_{v}^{r} F_{1,n-1}(x)dx \\ =\ & \int_{v}^{r} [F_{1,n-1}(r) - F_{1,n-1}(x)]dx \end{aligned}$$

Two cases:

i) $r > v$ (increasing bid above $b_{FP}(v)$). In this case, we have $F_{1,n-1}(r) - F_{1,n-1}(x) > 0,\ \forall x \in (v, r]$. Hence,

$$\int_{v}^{r} [F_{1,n-1}(r) - F_{1,n-1}(x)]dx > 0$$

ii) $r < v$ (decreasing bid below $b_{FP}(v)$). In this case, we have $F_{1,n-1}(r) - F_{1,n-1}(x) <$

$0, \forall x \in [r, v)$. Hence,

$$\begin{aligned} \int_v^r [F_{1,n-1}(r) - F_{1,n-1}(x)]dx &= \\ -\int_r^v [F_{1,n-1}(r) - F_{1,n-1}(x)]dx &> 0 \end{aligned}$$

Since $u(b_{FP}(v); v) - u(b_{FP}(v); v) = 0$, it follows that it is a unique best response for bidder 1 to bid $b_{FP}(v)$ when his valuation is v.

What's going on here?

We have shown that

$$b_{FP}(v) = \frac{\int_{\underline{v}}^v x f_{1,n-1}(x)dx}{F_{1,n-1}(v)}$$

To better interpret, we can rewrite this as (by partial integration)

$$\begin{aligned} b_{FP}(v) &= \frac{1}{F_{1,n-1}(v)}[[xF_{1,n-1}(x)]_{\underline{v}}^v - \int_{\underline{v}}^v F_{1,n-1}(x)dx] \\ &= v - \frac{\int_{\underline{v}}^v F_{1,n-1}(x)dx}{F_{1,n-1}(v)} < v \end{aligned}$$

So, each bidder shades his bid below his value. This is, of course, necessary to obtain a strictly positive payoff in case of winning - remember that in a FPSB auction, the winner pays his bid.

How much should the bid be shaded below the value? To answer this, we can use the comments on order statistics to rewrite $b_{FP}(v)$ in a slightly different manner

$$\begin{aligned} b_{FP}(v) &= \int_{\underline{v}}^v x \frac{f_{1,n-1}(x)}{F_{1,n-1}(v)} dx \\ &= \int_{\underline{v}}^v x f_{1,n-1}(x \mid V_{[1,n-1]} < v)dx \\ &= E(V_{[1,n-1]} \mid V_{[1,n-1]} < v) \end{aligned}$$

So, a bidder with valuation v bids the expectation of the highest of the other $n-1$ valuations, *conditioned* on this being less than v.

Finally, since $f_{1,n-1}(x \mid V_{[1,n-1]} < v) = f_{2,n}(x \mid V_{[1,n]} = v)$, we could alternatively write

$$\begin{aligned} b_{FP}(v) &= \int_{\underline{v}}^v x f_{2,n}(x \mid V_{[1,n]} = v)dx \\ &= E(V_{[2,n]} \mid V_{[1,n]} = v) \end{aligned}$$

to make things more clear.

So, for any given valuation v, the bidder proceeds on the assumption that it is the highest. Predicated on this, the bidder bids the expectation of the second highest valuation.

So, the bidder shades his bid below his value such that it coincides with his expectation of the valuation of his toughest competitor. ■

Example

Generalization of example in Gibbons (pp. 155-8), see also Jehle & Reny (Example 9.1).

Suppose $v_i \sim unif[0,1]$, $\forall i \rightarrow F(v) = v$ and $f(v) = 1$

$$\begin{aligned} b(v_i) &= v_i - \frac{\int_0^{v_i} x^{n-1} dx}{(v_i)^{n-1}} \\ &= v_i - \frac{1}{(v_i)^{n-1}} \left(\frac{1}{n}(v_i)^n\right) \end{aligned}$$

$$b(v_i) = v_i - \frac{1}{n} \frac{(v_i)^n}{(v_i)^{n-1}} \tag{2.18}$$

Hence,

$$b(v_i) = v_i - \frac{1}{n} v_i = \frac{n-1}{n} v_i \tag{2.19}$$

If $n = 2$, then $b(v_i) = \frac{1}{2} v_i$ as in Gibbons' special case. If $n \rightarrow \infty$, then $b(v_i) \rightarrow v_i$. We might think of the latter case as perfect competition between the bidders.

Strict monotonicity

We assumed above that $b(\cdot)$ was strictly increasing, and we used this when we inverted the bidding function. Can we show that the bidding function is, indeed, increasing?

$$\begin{aligned} \frac{db(v_i)}{dv_i} &= 1 - \frac{F^{n-1}(v_i) F^{n-1}(v_i) - (n-1) f(v_i) F^{n-2}(v_i) \int_{\underline{v}}^{v_i} F^{n-1}(x) dx}{(F^{n-1}(v_i))^2} \\ &= 1 - 1 + \frac{(n-1) f(v_i) F^{n-2}(v_i)}{(F^{n-1}(v_i))^2} \int_{\underline{v}}^{v_i} F^{n-1}(x) dx \\ &= \frac{(n-1) f(v_i) F^{n-2}(v_i)}{(F^{n-1}(v_i))^2} \int_{\underline{v}}^{v_i} F^{n-1}(x) dx > 0 \end{aligned}$$

Thus, $b(\cdot)$ is strictly increasing. Since $b(\cdot)$ is strictly increasing, the high-valuation bidder (with $v = V_{[1]}$) wins, and we conclude that the auction is efficient (in the sense that it puts the item in the hands of the bidder who values it the most).

2.4. Revenue equivalence (brute force)

Comparison of seller revenues in OAB/SPSB auctions and Dutch/FPSB auctions. We first use a "brute force approach". Below, we show things more elegantly (based on Riley & Samuelson (1981), Klemperer (1999) and Jehle & Reny (2000)).

a) *Expected revenues in OAB and SPSB auctions?*

In the SPSB auction, the winning bid is $V_{[1,n]}$, while the winning price and, hence, the revenue to the seller is $V_{[2,n]}$. It follows that the expected revenue in the SPSB auction is simply the unconditional expectation of the second-order statistic $V_{[2,n]}$:

$$\begin{aligned} ER_{SP} &= E\{V_{[2,n]}\} = \int_{\underline{v}}^{\overline{v}} x f_{2,n}(x)dx \\ &= \int_{\underline{v}}^{\overline{v}} n(n-1)xf(x)(1-F(x))F^{n-2}(x)dx \end{aligned}$$

b) *Expected revenues in Dutch and FPSB auctions?*

In the FPSB auction, the winning bid and, hence, the revenue to the seller is the maximum of $b_{FP}(v_i)$ accross the bidders, that is $b_{FP}(V_{[1,n]}) = E(V_{[2,n]} \mid V_{[1,n]} = v)$. Hence,

$$ER_{FP} = \int_{\underline{v}}^{\overline{v}} E(V_{[2,n]} \mid V_{[1,n]} = v) f_{1,n}(v)dv$$

given that $V_{[1,n]}$ is distributed according to $f_{1,n}(v)$. Using the derivations above we can replace $E(V_{[2,n]} \mid V_{[1,n]} = v)$ by $\int_{\underline{v}}^{v} x\frac{f_{1,n-1}(x)}{F_{1,n-1}(v)}dx$ to obtain

$$\begin{aligned} ER_{FP} &= \int_{\underline{v}}^{\overline{v}} (\int_{\underline{v}}^{v} x\frac{f_{1,n-1}(x)}{F_{1,n-1}(v)}dx) f_{1,n}(v)dv \\ &= \int_{\underline{v}}^{\overline{v}} (\int_{\underline{v}}^{v} x f_{1,n-1}(x)dx)\frac{f_{1,n}(v)}{F_{1,n-1}(v)}dv \\ &= \int_{\underline{v}}^{\overline{v}} (\int_{\underline{v}}^{v} x f_{1,n-1}(x)dx)\frac{nf(v)F^{n-1}(v)}{F^{n-1}(v)}dv \\ &= \int_{\underline{v}}^{\overline{v}} n(\int_{\underline{v}}^{v} x f_{1,n-1}(x)dx)f(v)dv \\ &= \int_{\underline{v}}^{\overline{v}} n(\int_{\underline{v}}^{v} x(n-1)f(x)F^{n-2}(x)dx)f(v)dv \\ &= \int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{v} n(n-1)xF^{n-2}(x)f(x)f(v)dxdv \end{aligned}$$

Change the order of integration to obtain

$$\begin{aligned}
ER_{FP} &= \int_{\underline{v}}^{\overline{v}} \int_{x}^{\overline{v}} n(n-1)xF^{n-2}(x)f(x)f(v)dvdx \\
&= \int_{\underline{v}}^{\overline{v}} n(n-1)xF^{n-2}(x)f(x)(\int_{x}^{\overline{v}} f(v)dv)dx \\
&= \int_{\underline{v}}^{\overline{v}} n(n-1)xF^{n-2}(x)f(x)(1-F(x)dx \\
&= \int_{\underline{v}}^{\overline{v}} n(n-1)xf(x)(1-F(x)F^{n-2}(x)dx \\
&= \int_{\underline{v}}^{\overline{v}} xf_{2,n}(x)dx = E\{V_{[2,n]}\} = ER_{SP}
\end{aligned}$$

Hence, we conclude that the *expected* revenues are the *same* in the *four standard auctions* in the IIPV case. This result known as

THE REVENUE EQUIVALENCE THEOREM (RET)

RET is one of the most important results in auction theory. It provides a bench-mark against which almost everything else is measured. Note the assumptions on which this is based, including risk neutrality, independence, symmetry and, of course, non-cooperative bidding. So, a key question for practical design is what happens to this result as assumptions are changed.

Interpretation and further comments

To get a better idea of what is going on here, we can rewrite the expected revenue in the standard auctions as

$$\begin{aligned}
ER &= E\{V_{[2,n]}\} \\
&= \int_{\underline{v}}^{\overline{v}} n(n-1)xf(x)(1-F(x))F^{n-2}(x)dx \\
&= \int_{\underline{v}}^{\overline{v}} nx(1-F(x))[(n-1)f(x)F^{n-2}(x)]dx \\
&= \int_{\underline{v}}^{\overline{v}} nx(1-F(x))\frac{d}{dx}[F^{n-1}(x)]dx
\end{aligned}$$

$$
\begin{aligned}
&= [nx(1-F(x))F^{n-1}(x)]_{\underline{v}}^{\overline{v}} - \int_{\underline{v}}^{\overline{v}} F^{n-1}(x)\frac{d}{dx}[nx(1-F(x))]dx \\
&= 0 - \int_{\underline{v}}^{\overline{v}} F^{n-1}(x)[n(1-F(x)) - nxf(x)]dx \\
&= \int_{\underline{v}}^{\overline{v}} xnf(x)F^{n-1}(x)dx - \int_{\underline{v}}^{\overline{v}} n(1-F(x))F^{n-1}(x)dx \\
&= \int_{\underline{v}}^{\overline{v}} xnf(x)F^{n-1}(x)dx - \int_{\underline{v}}^{\overline{v}} [\frac{1-F(x)}{f(x)}]nf(x)F^{n-1}(x)dx \\
&= \int_{\underline{v}}^{\overline{v}} xf_{1,n}(x)dx - \int_{\underline{v}}^{\overline{v}} [\frac{1-F(x)}{f(x)}]f_{1,n}(x)dx
\end{aligned}
$$

and we conclude that expected revenue in the standard auctions is

$$ER = E\{V_{[2,n]}\} = \int_{\underline{v}}^{\overline{v}} [v - \frac{1-F(v)}{f(v)}]f_{1,n}(v)dv \tag{2.20}$$

Below, the term in bracket $v - \frac{1-F(v)}{f(v)}$ will play an important role. Let us define

$$MR(v) = v - \frac{1-F(v)}{f(v)} \tag{2.21}$$

and rewrite (2.20) as

$$ER = \int_{\underline{v}}^{\overline{v}} MR(v)f_{1,n}(v)dv \tag{2.22}$$

What is $MR(v)$? See Klemperer (1999, App. B) and Jehle & Reny (2000, p. 395-).

Take an individual bidder with valuation distributed on $[\underline{v}, \overline{v}]$ according to $F(v)$ or $f(v) = F'(v)$.

[Fig. 1 about here - Figures are in the back]

Suppose a seller wants to sell the item to this bidder by simply posting some price $p = \widehat{v}$. Then we can interpret $1 - F(\widehat{v})$ as the bidder's demand function. With a price of $\widehat{v}$, the bidder will buy the unit if his valuation exceeds $\widehat{v}$, which is the case with probability $1 - F(\widehat{v})$. So, by setting $p = \widehat{v}$ the monopolist's expected quantity of sales will be $q(\widehat{v}) = 1 - F(\widehat{v})$. Now, in usual fashion, define the marginal revenue as

$$MR(v) = \frac{d(v \cdot q(v))}{dq} = v + q\frac{dv}{dq} = v + \frac{q}{dq/dv}$$

Now, $q = q(v) = 1 - F(v)$, which implies that $\frac{dq}{dv} = -f(v)$, and we can write the marginal revenue as

$$MR(v) = v + \frac{1 - F(v)}{-f(v)} = v - \frac{1 - F(v)}{f(v)}$$

Intuitively, think of this as the marginal revenue to the seller of a bidder with valuation v. In (2.22), note that $f_{1,n}(v) = nf(v)F^{n-1}(v)$ is the density of the highest valuation $V_{[1]}$, and we conclude that

$$ER = \text{ Expected } MR \text{ of the winning bidder}$$

Heuristics

For a given value ("price") $\widehat{v}$ we have

$$\begin{aligned}
\widehat{v}q(\widehat{v}) &= \int_{q=0}^{q(\widehat{v})} MR(v(q))dq = \int_{q=0}^{q(\widehat{v})} MR(v(q))\frac{dq}{dv}\frac{dv}{dq}dq \\
&= \int_{q=0}^{q(\widehat{v})} MR(v(q))\frac{d(1-F(v))}{dv}\frac{dv}{dq}dq \\
&= \int_{q=0}^{q(\widehat{v})} MR(v(q))(-f(v(q)))dv = -\int_{\overline{v}}^{\widehat{v}} MR(v)f(v)dv \\
&= \int_{\widehat{v}}^{\overline{v}} MR(v)f(v)dv
\end{aligned}$$

Consider an OAB auction, and let $\widehat{v}$ be the actual value of the second-highest bidder ($\widehat{v} = V_{[2,n]}$). Then $p_{OAB} = \widehat{v}$, and

$$\widehat{v} = \frac{1}{q(\widehat{v})}\int_{q=0}^{q(\widehat{v})} MR(v(q))dq = \frac{1}{1 - F(\widehat{v})}\int_{\widehat{v}}^{\overline{v}} MR(v)f(v)dv$$

Now, suppose that bidder i is the winner, that is, $v_i = V_{[1,n]} > \widehat{v}$. Then, $\widehat{v}$ equals the average value of bidder i's MR *conditional on* $v_i > \widehat{v}$.

[Fig. 2 about here]

All we know about bidder i's valuation is $v_i > \widehat{v}$ since he has won. This implies that for any actual second-highest value $\widehat{v}$, the auction price (= the actual revenue) coincides with the expected MR of the winning bidder!!

This result, of course, carries over to *any auction which is revenue equivalent* to the OAB auction. Under the stated assumptions, we conclude that

The expected revenue from any standard auction equals the expected marginal revenue of the winning bidder

Example (continued from earlier)
$v_i \sim unif[0,1] \rightarrow \underline{v} = 0,\ \overline{v} = 1,\ F(v) = v,\ f(v) = 1$

$$\begin{aligned} ER &= \int_0^1 MR(v) f_{1,n}(v)dv \\ &= \int_0^1 [v - \tfrac{1-F(v)}{f(v)}] n f(v) F^{n-1}(v)dv \\ &= n\int_0^1 [v - 1 + v] v^{n-1} dv \\ &= n\int_0^1 [2v-1] v^{n-1} dv \\ &= 2n\int_0^1 v^n dv - n\int_0^1 v^{n-1} dv \\ &= 2n[\tfrac{1}{n+1} v^{n+1}]_0^1 - n[\tfrac{1}{n} v^n]_0^1 \\ &= \tfrac{2n}{n+1} - \tfrac{n}{n} \end{aligned}$$

or

$$ER = \frac{n-1}{n+1}$$

E.g., $n = 2 \Rightarrow ER = \frac{1}{3}$ and $n \rightarrow \infty \Rightarrow ER \rightarrow 1$.

Remarks on standard auctions

- Expected revenues coincide, but actual revenues may differ. This would be a concern for the seller, if he was risk averse.
- Efficiency: Yes, in all the standard auctions, the high valuation bidder obtains the item for sure. Note that we have (implicitly) assumed so far that the item is of no value to the seller.
- Increasing n (the number of bidders). $E\{V_{[2,n]}\}$ is increasing in n which implies that the expected revenue is increasing in the number of bidders. Note that $E\{V_{[2,n]}\} \rightarrow \overline{v}$ for $n \rightarrow \infty$.

What about optimality from the point of view of the seller? Is it possible to do better than the standard auctions? We study this below based on Riley and Samuelson (1981), Klemperer (1999) and Jehle & Reny (2000). But first, we derive RET more elegantly.

2.5. Revenue equivalence (elegantly)

Based on Klemperer (1999, App. A). Take any allocation mechanism. In equilibrium this gives rise to

- $S_i(v)$ - the expected surplus of bidder i from participating if his type is v.

- $P_i(v)$ - the probability that bidder i "wins" if his type is v.
- $E(v)$ - the expected payment of bidder i if his type is v.

Using these definitions, we can write

$$S_i(v) = vP_i(v) - E(v)$$

Now, bidder i of type v should have an incentive to play as if he is indeed of type v rather than some other type $\widetilde{v} \neq v$. The payoffs from playing as if v are

$$S_i(v) = vP_i(v) - E(v)$$

while the payoffs from playing as if $\widetilde{v}$ are

$$\begin{aligned} S_i(\widetilde{v}; v) &= vP_i(\widetilde{v}) - E(\widetilde{v}) \\ &= \widetilde{v}P_i(\widetilde{v}) - \widetilde{v}P_i(\widetilde{v}) + vP_i(\widetilde{v}) - E(\widetilde{v}) \\ &= \widetilde{v}P_i(\widetilde{v}) - E(\widetilde{v}) + (v - \widetilde{v})P_i(\widetilde{v}) \\ &= S_i(\widetilde{v}) + (v - \widetilde{v})P_i(\widetilde{v}) \end{aligned}$$

For incentive-compatibility, we require for type v that

$$S_i(v) \geq S_i(\widetilde{v}; v), \ \forall \widetilde{v}$$

which we write as

$$S_i(v) \geq S_i(\widetilde{v}) + (v - \widetilde{v})P_i(\widetilde{v}), \ \forall \widetilde{v} \tag{2.23}$$

Similarly, for the actual type $\widetilde{v}$ we require

$$S_i(\widetilde{v}) \geq S_i(v; \widetilde{v}), \ \forall v$$

which we write as

$$S_i(\widetilde{v}) \geq S_i(v) + (\widetilde{v} - v)P_i(v), \ \forall v \tag{2.24}$$

Now, let $\widetilde{v} = v + dv$ (where dv is positive and small). Then, (2.23) becomes

$$S_i(v) \geq S_i(v + dv) + (v - v - dv)P_i(v + dv)$$

or

$$S_i(v) \geq S_i(v + dv) + (-dv)P_i(v + dv) \tag{2.25}$$

Similarly, (2.24) becomes

$$S_i(v + dv) \geq S_i(v) + (v + dv - v)P_i(v)$$

or

$$S_i(v + dv) \geq S_i(v) + dvP_i(v) \tag{2.26}$$

Rearranging (2.25) and (2.26) gives

$$P_i(v + dv) \geq \frac{S_i(v + dv) - S_i(v)}{dv} \geq P_i(v)$$

Take the limit as $dv \to 0$ (assuming existence), and we get

$$\frac{dS_i(v)}{dv} = P_i(v) \tag{2.27}$$

The left-hand-side is the derivative of the expected surplus to a bidder with valuation v, while the right-hand-side is the probability of winning. From (2.27) we get

$$S_i(v) = S_i(\underline{v}) + \int_{\underline{v}}^{v} P_i(v)dv \tag{2.28}$$

which is illustrated in Fig. 3.

[Fig. 3 about here]

Digression

Connect Jehle & Reny (9.3.1) and Klemperer (App. A). In the notation of Jehle & Reny

- $p_i(\cdot)$ - the probability assignment in the mechanism, that is, the probability with which bidder i obtains the item.
- $c_i(\cdot)$ - the cost function, that is, the transfer from bidder i to the mechanism (the seller).
- $U_i(\cdot)$ - the expected surplus of bidder i.
- $\underline{v}= 0$, $\overline{v} = 1$.

Jehle & Reny state the following result.

Theorem 2.1 (Jehle & Reny 9.5). *Incentive Compatible Direct Mechanism*

A direct selling mechanism $(p_i(\cdot), c_i(\cdot))_{i=1}^{n}$ is incentive-compatible if and only if for every bidder i

(i) $\overline{p}_i(v_i)$ is non-decreasing in v_i, and

(ii) $\overline{c}_i(v_i) = \overline{c}_i(0) + \overline{p}_i(v_i)v_i - \int_0^{v_i} \overline{p}_i(x)dx$, for every $v_i \in [0, 1]$.

Proof: See Jehle & Reny (pp. 386-7). ■

Now, let

$$\overline{p}_i(r_i) = \int_0^1 \int_0^1 .... \int_0^1 p_i(r_i; v_{-i}) f_{-i}(v_{-i}) dv_{-i}$$

and

$$\overline{c}_i(r_i) = \int_0^1 \int_0^1 .... \int_0^1 c_i(r_i; v_{-i}) f_{-i}(v_{-i}) dv_{-i}$$

We use these to write the surplus to bidder i from pretending to have valuation r_i as

$$u_i(r_i, v_i) = \overline{p}_i(r_i) v_i - \overline{c}_i(r_i)$$

With truth-telling (incentive-compatibility) we have

$$U_i(v_i) = u_i(v_i, v_i) = \overline{p}_i(v_i) v_i - \overline{c}_i(v_i)$$

We can rewrite (ii) in the theorem as follows

$$\overline{c}_i(v_i) = \overline{c}_i(0) + \overline{p}_i(v_i) v_i - \int_0^{v_i} \overline{p}_i(x) dx$$

$$\Leftrightarrow \overline{p}_i(v_i) v_i - \overline{c}_i(v_i) = -\overline{c}_i(0) + \int_0^{v_i} \overline{p}_i(x) dx$$

$$\Leftrightarrow \overline{p}_i(v_i) v_i - \overline{c}_i(v_i) = \overline{p}_i(0) \cdot 0 - \overline{c}_i(0) + \int_0^{v_i} \overline{p}_i(x) dx$$

$$U_i(v_i) = U_i(0) + \int_0^{v_i} \overline{p}_i(x) dx$$

But in the terminology of Klemperer (1999, App. A) this is just

$$S_i(v_i) = S_i(\underline{v}) + \int_{\underline{v}}^{v_i} P_i(x) dx$$

which has to hold in any incentive-compatible mechanism, in addition to $P_i(x)$ being increasing (cf. Fig. 3 above).

End of digression - return to main argument!

Where does this take us

Take *any* two mechanisms for allocating the item with the *same* $S_i(\underline{v})$ (*surplus at low end*) and the *same* $P_i(v)$ (*probability of winning*) functions. Then the mechanisms have the same $S_i(v)$ functions. This in turn implies that $E(v)$ (the expected payment of bidder i of type v) is the same in the two mechanisms. Formally, take two mechanisms such that

$$S_i^1(v) = vP_i^1(v) - E^1(v)$$

and

$$S_i^2(v) = vP_i^2(v) - E^2(v)$$

If

$$P_i^1(v) = P_i^2(v)$$

and

$$S_i^1(\underline{v}) = S_i^2(\underline{v})$$

then

$$S_i^1(v) = S_i^2(v)$$

Hence,

$$S_i^1(v) = S_i^2(v) \text{ and } P_i^1(v) = P_i^2(v)$$

and it follows that

$$E^1(v) = E^2(v)$$

In other words, every type of every player makes the same expected payment in both mechanisms. This evidently implies that the two mechanisms generate the same expected revenue to the seller. We can state the following version of the Revenue Equivalence Teorem.

> RET (IIPV): Any auction mechanism which (i) always awards the item to the bidder with the highest valuation, and (ii) gives the same (e.g. zero) surplus to the bidder with the lowest possible valuation, yields the *same* expected revenue and results in each bidder making the *same* expected payment as a function of his valuation.

Other formulations are possible (see Klemperer (1999).

Significance for standard auctions?

(OAB, Dutch, FPSB, SPSB and simple all-pay formats (see below))

If bidding strategies are increasing in v, if the object is awarded to the highest bidder, and if $S_i(\underline{v}) = 0$, then

$$P_i(v) = F^{n-1}(v)$$

Further

$$S_i(v) = S_i(\underline{v}) + \int_{\underline{v}}^{v} P_i(x)dx = \int_{\underline{v}}^{v} F^{n-1}(x)dx$$

which is the same for all, and we have RET. We conclude directly from this that the standard auctions are revenue equivalent (if IIPV).

Multiple objects?

Arguments leading to RET did not rely on there being a *single* object on offer. If more than one object *and* if each bidder desires at most one object, then RET still applies in the IIPV case (see Klemperer (1999, p. 44).

2.6. Bidding strategies (again)

We can use the RET to compute bidding strategies much more easily than we did above. To see this, start from the OAB auction, which satisfies the conditions for RET. In the OAB auction, it is optimal to stay in the bidding until $p = v$. Hence

$$b(v) = v$$

Further, $E(v)$ (the expected payment) is the probability of winning ($P_i(v) = F^{n-1}(v)$) *times* the expectation of the *highest* of the remaining $n-1$ bids/valuati-ons, *conditional* on these being below v. This expectation is

$$v - \frac{\int_{\underline{v}}^{v} F^{n-1}(x)dx}{F^{n-1}(v)}$$

as we have already seen. This implies that the expected payment is

$$E(v) = F^{n-1}(v)(v - \frac{\int_{\underline{v}}^{v} F^{n-1}(x)dx}{F^{n-1}(v)}) = F^{n-1}(v)v - \int_{\underline{v}}^{v} F^{n-1}(x)dx \tag{2.29}$$

in all auction formats that satisfies RET.

FPSB Auction: In a FPSB auction, type v's expected payments are $P_i(v) = F^{n-1}(v)$ times his bid $b(v)$. That is,

$$E(v) = F^{n-1}(v)b(v)$$

and (from (2.29))

$$E(v) = F^{n-1}(v)v - \int_{\underline{v}}^{v} F^{n-1}(x)dx$$

Combining these two expression for the expected payment, we immediately obtain the bidding strategy as

$$b_{FP}(v) = v - \frac{\int_{\underline{v}}^{v} F^{n-1}(x)dx}{F^{n-1}(v)}$$

Note how easy this derivation was.

Simple All-Pay Auction: By this we mean that the highest bidder wins the object, and *all* bidders pay their bid.

When all bidders pay their bid, the expected payments of type v are simply the bid $b(v)$. That is,

$$E(v) = b(v)$$

and from (2.29)

$$E(v) = F^{n-1}(v)v - \int_{\underline{v}}^{v} F^{n-1}(x)dx$$

Combining these two expressions for the expected payments, we immediately obtain the bidding function in the all-pay auction as

$$b_{AP}(v) = F^{n-1}(v)v - \int_{\underline{v}}^{v} F^{n-1}(x)dx$$

Note that this is just

$$b_{AP}(v) = F^{n-1}(v)b^{FP}(v) < b^{FP}(v)$$

2.7. Optimal auctions

Based on Riley & Samuelson (1981) with alternative expositions in Myerson (1981), Jehle & Reny (2000, ch. 9) and Krishna (2002).

Assumptions

- Seller with valuation v_0 faces n potential buyers.
- Buyer-side satisfies IIPV (independently and identically distributed valuations), $F(v)$, $F(\underline{v}) = 0$, $F(\overline{v}) = 1$, $F(\cdot)$ strictly increasing and differentiable on $[\underline{v}, \overline{v}]$, $F'(\cdot) = f(\cdot)$.

The family of auction rules $\Re$

(1) A bidder can make any bid above some minimum set by the seller.
(2) The highest bidder (above the minimum) wins.
(3) The auction rules are anonymous (non-discrimination).
(4) There is assumed to exist a common equilibrium bidding strategy in which $b_i = b(v_i)$, $i = 1, 2, 3, ..., n$, is strictly increasing.

Thus we have enlarged to scope of the seller: The seller may demand a minimum or reserve price, and losing bidders may be required to pay.

Absent bidder collusion, we look for BNE of various mechanisms. Without loss of generality, we focus on bidder 1. We consider the choice of b_1 given the hypothesis that all others bid according to $b(v)$. Define the following bid interval

$$B = [b(\underline{v}), b(\overline{v})]$$

Bidder 1 will never bid outside B. So, essentially, bidder 1's choice of b_1 is a choice of x such that $b_1 = b(x)$ (this should be familiar by now). Then, $b(x)$ is a BNE bidding strategy, if bidder 1 can do no better than choose $x = v_1$, and it follows that $b_1 = b(v_1)$.

Mechanism/Auction Rule

(1) Payment from bidder 1 as a function of all bids

$$\begin{aligned} p &= p(b_1, b_2, ..., b_n) \\ &= p(b(x), b(v_2), ..., b(v_n)) \end{aligned}$$

(2) Bidder 1 wins if he is the highest bidder (if above the minimum).

Bidder 1's expected payment from choosing x is

$$P(x) = E_{v_2,.....,v_n}\{p(b(x), b(v_2), ..., b(v_n))\}$$

and the expected return/surplus to bidder 1 from bidding $b(x)$ when his true valuation is v_1 is

$$u(x, v_1) = v_1 F^{n-1}(x) - P(x)$$

For $b(v)$ to be a best response (BNE), it must be the case that

$$\frac{\partial u(x, v_1)}{\partial x} = v_1 \frac{dF^{n-1}(x)}{dx} - P'(x) = 0 \text{ at } x = v_1 \tag{2.30}$$

This must hold for all $v_1 \geq v_*$. v_* is the valuation for which bidder 1 is indifferent between submitting the bid $b(v_*)$ and not entering the auction at all. That is, v_* is defined by

$$u(v_*, v_*) = v_* F^{n-1}(v_*) - P(v_*) = 0$$

or

$$P(v_*) = v_* F^{n-1}(v_*)$$

We use (2.30) to write

$$P'(v_1) = v_1 \frac{dF^{n-1}(v_1)}{dv_1}$$

From this differential equation, we obtain for $v_1 \geq v_*$

$$\begin{aligned} P(v_1) &= P(v_*) + \int_{v_*}^{v_1} P'(x)dx \\ &= v_* F^{n-1}(v_*) + \int_{v_*}^{v_1} x \frac{dF^{n-1}(x)}{dx} dx \\ &= v_* F^{n-1}(v_*) + [x F^{n-1}(x)]_{v_*}^{v_1} - \int_{v_*}^{v_1} F^{n-1}(x)dx \end{aligned}$$

or

$$P(v_1) = v_1 F^{n-1}(v_1) - \int_{v_*}^{v_1} F^{n-1}(x)dx$$

The seller receives the expected value of $P(v_1)$ from bidder 1 which is

$$\begin{aligned}
\overline{p}^1 &= \int_{v_*}^{\overline{v}} P(v_1)f(v_1)dv_1 \\
&= \int_{v_*}^{\overline{v}} [v_1 F^{n-1}(v_1) - \int_{v_*}^{v_1} F^{n-1}(x)dx] f(v_1)dv_1 \\
&= \int_{v_*}^{\overline{v}} v_1 F^{n-1}(v_1)f(v_1)dv_1 - \int_{v_*}^{\overline{v}} f(v_1)(\int_{v_*}^{v_1} F^{n-1}(x)dx)dv_1 \\
&= \int_{v_*}^{\overline{v}} vF^{n-1}(v)f(v)dv - [F(v)\int_{v_*}^{v} F^{n-1}(x)dx]_{v_*}^{\overline{v}} + \int_{v_*}^{\overline{v}} F(v)F^{n-1}(v)dv \\
&= \int_{v_*}^{\overline{v}} vF^{n-1}(v)f(v)dv - \int_{v_*}^{\overline{v}} F^{n-1}(v)dv + \int_{v_*}^{\overline{v}} F(v)F^{n-1}(v)dv \\
&= \int_{v_*}^{\overline{v}} [vf(v) - (1-F(v))]F^{n-1}(v)dv \\
&= \frac{1}{n}\int_{v_*}^{\overline{v}} [v - \frac{(1-F(v))}{f(v)}]nf(v)F^{n-1}(v)dv
\end{aligned}$$

Since there are n ex ante symmetric bidders, the total expected payments to the seller are

$$\begin{aligned}
n\overline{p}^1 &= \int_{v_*}^{\overline{v}} [v - \frac{(1-F(v))}{f(v)}]nf(v)F^{n-1}(v)dv \\
&= \int_{v_*}^{\overline{v}} [v - \frac{(1-F(v))}{f(v)}]f_{1,n}(v)dv \\
&= \int_{v_*}^{\overline{v}} MR(v)f_{1,n}(v)dv
\end{aligned}$$

So, slightly reformulating Theorem 1 in Riley & Samuelson (1981), we have

Theorem 2.2. *Assume IIPV and bidder risk neutrality. Then, the common equilibrium bidding strategy for every member of family $\Re$ gives*

$$ER = \int_{v_*}^{\overline{v}} [v - \frac{(1-F(v))}{f(v)}]nf(v)F^{n-1}(v)dv = \int_{v_*}^{\overline{v}} MR(v)f_{1,n}(v)dv$$

where v_* is the bidder valuation below which it is unprofitable to bid.

Remarks. Looks familiar in light of what we saw in the previous section.

- Expected revenues from different auction formats can be compared simply by finding the lowest bidder value for which it is worthwhile to enter the auction and become an active bidder. Also, auctions with the same v_* generate the same expected revenue to the seller.
- Note that nowhere in the derivations did we refer explicitly to equilibrium bidding strategies. However, if we are given some mechanism/auction in the family $\Re$, it is straightforward to derive $b(v)$.

Bidding strategies

Consider a FPSB auction with a minimum bid/reserve price b_0 set by the seller.

a) Any bidder with $v > b_0$ has an incentive to enter a bid, and

$$v_* = b_0$$

Bidders with valuations below b_0 enter no bids.

b) In a FPSB auction, a bidder only pays if he wins, and then he pays his bid $b(v)$. Hence, the expected payment as a function of v is

$$\begin{aligned} P(v) &= pr\{b(v) \text{ is the winning bid}\} \times b(v) \\ &= F^{n-1}(v)b(v) \end{aligned}$$

Given $v_* = b_0$ we also have

$$P(v) = vF^{n-1}(v) - \int_{b_0}^{v} F^{n-1}(x)dx$$

Combining the two expressions for the expected payment, we obtain

$$b_{FP}(v) = v - \frac{\int_{b_0}^{v} F^{n-1}(x)dx}{F^{n-1}(v)}$$

Except that b_0 replaces $\underline{v}$, this is the same as in the standard FPSB auction without a reserve price. Riley and Samuelson state the following result.

Theorem 2.3. *Assume IIPV, risk neutrality, and a reserve price* b_0. *Then, in the Dutch/FPSB auction, the equilibrium bidding strategy for* $v \geq b_0$ *is*

$$b_{FP}(v) = v - \frac{\int_{b_0}^{v} F^{n-1}(x)dx}{F^{n-1}(v)}$$

Remarks

- $b_{FP}(v)$ is increasing, and it follows that Dutch/FPSB auctions with a reserve price belong to the family $\Re$.
- OAB/SPSB auctions with a reserve price clearly also belong to the family $\Re$. Essentially, it is optimal to bid one's value, that is, $b(v) = v$, which is increasing. Further, bidders want to enter a bid if and only if $v \geq b_0 = v_*$.

- We conclude that Dutch, FPSB, OAB and SPSB auctions with a reserve price all belong to the family $\Re$, and they generate the same $v_* = b_0$.

Corollary 1. *The standard auctions with a reserve price b_0 generate the same expected revenue to the seller.*

So, yet again, we have proved RET rather more elegantly than we first did (both for $b_0 = 0$ and $b_0 > 0$).

Optimal Auctions

The remaining question is: which auctions in $\Re$ are optimal from the point of view of the seller? Given the first theorem of Riley & Samuelson (1981), this is easy to answer.

a) Given an implied minimum valuation for bidder activity, v_*, the expected revenue of the seller is

$$ER = \int_{v_*}^{\overline{v}} [v - \frac{(1 - F(v))}{f(v)}] n f(v) F^{n-1}(v) dv$$

b) If the object is not sold, then the seller receives his valuation v_0, and the probability of no sale is $F^n(v_*)$ (i.e., the probability that all bidders have valuations less than v_*).

Let us combine a) and b) to write the expected payoff (*not revenue*) to the seller in an auction format that induces v_* as

$$v_0 F^n(v_*) + \int_{v_*}^{\overline{v}} [v - \frac{(1 - F(v))}{f(v)}] n f(v) F^{n-1}(v) dv$$

The objective of the seller is to maximize this with respect to v_* (resolve below how we implement the optimal v_*).

First-order condition

$$v_0 n f(v_*) F^{n-1}(v_*) - [v_* - \frac{1 - F(v_*)}{f(v_*)}] n f(v_*) F^{n-1}(v_*) = 0$$

or

$$v_0 = v_* - \frac{1 - F(v_*)}{f(v_*)}$$

This looks very familiar. Recall that

$$v_* - \frac{1 - F(v_*)}{f(v_*)} = MR(v_*)$$

and think of v_0 as "the marginal cost" of the seller, MC. Then, v_* should solve

$$MC = MR(v_*)$$

So, in order to maximize payoffs, the seller wants to "exclude" bidders with MR's below his own MC. This is illustrated in Fig. 4.

[Fig. 4 about here]

Remark. There may be multiple v_*'s that solve the first-order condition. The first-order condition is only necessary, and to find the optimum, we must compare the payoffs of the seller at the different solutions.

Second-order condition:

We can write the first-order condition as

$$nF^{n-1}(v_*)[v_0 f(v_*) - v_* f(v_*) + (1 - F(v_*))] = 0$$

Hence

$$(v_0 - v_*)f(v_*) + (1 - F(v_*)) = 0 \tag{2.31}$$

and

$$v_0 - v_* = -\frac{1 - F(v_*)}{f(v_*)} \tag{2.32}$$

Differentiate the first-order condition to get the second-order condition

$$\begin{array}{ll} n(n-1)f(v_*)F^{n-2}(v_*)[(v_0 - v_*)f(v_*) + (1 - F(v_*))] & + \\ nF^{n-1}(v_*)[(v_0 - v_*)\frac{df(v_*)}{dv_*} - f(v_*) - f(v_*)] & < 0 \end{array}$$

Using (2.31), the first term is zero, and we get

$$(v_0 - v_*)\frac{df(v_*)}{dv_*} - 2f(v_*) < 0$$

and using (2.32) we can write this as

$$-\frac{1 - F(v_*)}{f(v_*)}\frac{df(v_*)}{dv_*} - 2f(v_*) < 0$$

or

$$(1 - F(v_*))\frac{df(v_*)}{dv_*} + 2[f(v_*)]^2 > 0$$

It follows that a *sufficient* condition for v_* solving the first-order condition to be a global, interior maximum is

$$(1 - F(v))\frac{df(v)}{dv} + 2[f(v)]^2 > 0,\ \forall v \in [\underline{v}, \overline{v}]$$

Now, recall that

$$MR(v) = v - \frac{1 - F(v)}{f(v)}$$

hence,

$$\begin{aligned} MR'(v) = \frac{dMR(v)}{dv} &= 1 - \frac{f(v)(-f(v)) - (1-F(v))\frac{df(v)}{dv}}{[f(v)]^2} \\ &= \frac{1}{[f(v)]^2}[[f(v)]^2 + [f(v)]^2 + (1 - F(v))\frac{df(v)}{dv}] \\ &= \frac{1}{[f(v)]^2}[(1 - F(v))\frac{df(v)}{dv} + 2[f(v)]^2] \end{aligned}$$

Now, $MR'(v) > 0$ if and only if

$$(1 - F(v))\frac{df(v)}{dv} + 2[f(v)]^2 > 0$$

But this is basically our second-order condition.

Conclusions

- $MR(v)$ increasing for all $v \in [\underline{v}, \overline{v}]$ is a sufficient condition for a global, interior maximum at v_*.
- $MR(v)$ increasing is exactly the assumption stressed by both Klemperer (1999, p. 50) and Jehle & Reny (2000, p. 395-).

Returning to

$$v_0 = v_* - \frac{1 - F(v_*)}{f(v_*)} = MR(v_*)$$

and the interpretation that the seller wants to exclude bidders with $MR(v)$'s less than v_0 (= "MC"), we can define $p_i^*(v_1, v_2, ..., v_n)$ as the probability that bidder i wins. Then

$$p_i^*(v_1, v_2, ..., v_n) = \begin{cases} 1 & \text{if } MR(v_i) > \max\{v_0, MR(v_j)\}, \forall j \neq i \\ 0 & \text{otherwise} \end{cases}$$

in the optimal auction. Rephrasing Theorem 3 in Riley & Samuelson (1981), we can state.

Theorem 2.4. *Assume IIPV, bidder risk neutrality, and increasing $MR(v)$. Then, auction rules in $\Re$ which maximize expected seller* ***payoffs****, will have a reservation value for bidder activity, v_*, which satisfies*

$$v_* = v_0 + \frac{1 - F(v_*)}{f(v_*)}$$

independently of the number of bidders.

Remarks on optimal auctions

a) The critical bidder reservation value exceeds the seller's own valuation of the object, $v_* > v_0$.

To see this, note that we have

$$MR(v_*) = v_* - \frac{1 - F(v_*)}{f(v_*)} = v_0 + \frac{1 - F(v_*)}{f(v_*)} - \frac{1 - F(v_*)}{f(v_*)} = v_0$$

Hence

$$v_* = MR^{-1}(v_0) > v_0$$

But this is just like the monopolist in the standard text book who sets a price larger than marginal costs (v_0). From this is is immediate that

> Dutch, FPSB, OAB and SPSB auctions are optimal, provided that the seller announces a reserve price/minimum bid, b_0, defined as
>
> $$b_0 = v_* = MR^{-1}(v_0) > v_0$$

b) Optimal auctions from the point of view of the seller may induce Pareto-*in*efficient allocations.

> If $V_{[1,n]} \in (v_0, v_*)$, then the object will not be traded, even though there are strictly positive gains from trade, $V_{[1,n]} > v_0$.

This is not surprising, since the seller is, after all, a monopolist.

Further intuition on optimal auctions

I) Why optimal to set reserve above v_0?

We *trade off* increases in the selling price if there is a trade *against* increases in the probability that there will be no trade.

Consider the OAB auction: it is dominant to stay in the bidding as long as $p < v$.

- Disadvantage of $b_0 > v_0$ is that $V_{[1,n]} \in (v_0, b_0)$ has positive probability.
- Advantage of $b_0 > v_0$ is that in case $V_{[1,n]} \geq b_0 > \max\{V_{[2,n]}, v_0\}$, the selling price rises from $\max\{V_{[2,n]}, v_0\}$ to b_0.

Illustration: Suppose $n = 1$ (one bidder), $v_0 = 0$, and $v_1 \sim unif[0,1]$.
With optimal reserve price

$$b_0 = v_* = v_0 + \frac{1 - F(v_*)}{f(v_*)} = 0 + 1 - v_* = v_* = \frac{1}{2}$$

and

$$ER = \{\text{price if sale}\} \times \{\text{probability of sale}\} = \{\frac{1}{2}\} \times \{\frac{1}{2}\} = \frac{1}{4}$$

Without reserve, the selling price is 0 (no matter what), and we get

$$ER = 0$$

II) The critical bidder valuation, v_*, is independent of the number of bidders, n.

This can be understood by reference to the marginal revenues of bidders. Refer to 3rd degree price discrimination by monopolist (only wants to serve those who contribute more than marginal costs, and wants to serve *all* these).

III) In the four standard auctions, the optimal reserve price, b_0, is independent of the number of bidders, n. Not necessarily so in "non-standard" auctions, such as the all-pay auction (see below).

The purpose of the reserve price is to cut into the *informational rents* of the winner, and these rents are

$$\frac{1 - F(v)}{f(v)}$$

independently of n. To see this more clearly, we can rewrite the expected seller payoffs as

$$\begin{aligned} & v_0 F^n(v_*) + \int_{v_*}^{\overline{v}} [v - \tfrac{(1-F(v))}{f(v)}] n f(v) F^{n-1}(v) dv \\ = \; & v_0 F^n(v_*) + \int_{v_*}^{\overline{v}} v n f(v) F^{n-1}(v) dv - \int_{v_*}^{\overline{v}} [\tfrac{(1-F(v))}{f(v)}] n f(v) F^{n-1}(v) dv \end{aligned}$$

In the last expression, the first two terms give the total value of exchange. The last term, which is subtracted, captures the payoffs to buyers, that is, payoffs in excess of valuation or *rent*.

We differentiate the whole expression with respect to v_* and set to zero. That is,

$$\begin{aligned} v_0 n f(v_*) F^{n-1}(v_*) - v_* n f(v_*) F^{n-1}(v_*) &= -\tfrac{1-F(v_*)}{f(v_*)} n f(v_*) F^{n-1}(v_*) \\ -(v_* - v_0) n f(v_*) F^{n-1}(v_*) &= -\tfrac{1-F(v_*)}{f(v_*)} n f(v_*) F^{n-1}(v_*) \\ -(v_* - v_0) f^*(v_*) &= -\tfrac{1-F(v_*)}{f(v_*)} f^*(v_*) \end{aligned}$$

Now, the left-hand-side is the decrease in total value of exchange from increasing v_* above v_0, while the right-hand-side is the decrease in rents from increasing v_* above v_0.

Note how the number of bidders, n, cancels out. It has the same effect on value and rents at optimum.

IV) Introducing a reserve price works like adding a player bidding b_0.

This, of course, is most likely to affect the outcome of an auction when the number of bidders is low. With many bidders, the reserve is less likely to bind.

V) Entry fees constitute an alternative way to induce v_*.

Definitions:

C - fixed entry/participation fee paid by all active bidders.

v_c - minimum valuation for which it pays to participate (and incur the cost C for sure).

Illustration: Take an SPSB auction with an entry fee C. Consider a bidder with valuation v_c. Now, after entry, C is a sunk cost, and it is optimal to bid v_c. The bidder with valuation v_c wins if there are no other active bidders - other active bidders will have $v > v_c$ and will bid more than v_c. To be indifferent, the expected payoff to the bidder with the critical valuation v_c must be equal to zero, that is

$$v_c F^{n-1}(v_c) - C = 0$$

Now, the seller must choose C optimally. From the point of view of the seller, we have that the minimum valuation active bidder, v_*, should be indifferent between participating and not participating. Hence, the optimal C_*, which implies v_*, is defined by

$$v_* F^{n-1}(v_*) - C = 0$$

or

$$C_* = v_* F^{n-1}(v_*) \ (= (v_0 + \frac{1 - F(v_*)}{f(v_*)}) F^{n-1}(v_*))$$

It follows that an SPSB auction *without* a reserve price *but with* and entry fee, C_*, is optimal, where C_* and v_* solve

$$\begin{aligned} v_* &= v_0 + \tfrac{1-F(v_*)}{f(v_*)} \\ C_* &= v_* F^{n-1}(v_*) \end{aligned}$$

Example: $v \sim unif[0,1]$, $F(v) = v$ and $f(v) = 1$

$$v_* = v_0 + \frac{1 - F(v_*)}{f(v_*)} = v_0 + 1 - v_* \Rightarrow v_* = \frac{1 + v_0}{2}$$

$$C_* = v_* F^{n-1}(v_*) = v_*(v_*)^{n-1} = (v_*)^n \Rightarrow C_* = \left(\frac{1 + v_0}{2}\right)^n$$

Note that the entry fee, C_*, depends on the number of bidders! (decreasing in n)

2.8. Examples (Optimality of auction formats)

We investigate the optimality of different auction formats by considering a series of examples.

2.8.1. Uniform distribution of valuations - general

There are n bidders, bidder valuations are distributed as $v_i \sim unif[\underline{v}, \overline{v}]$, for the seller $v_0 \in [\underline{v}, \overline{v}]$ and this value is known by all. We have the following

$$F(v) = \frac{v - \underline{v}}{\overline{v} - \underline{v}}, f(v) = \frac{1}{\overline{v} - \underline{v}}$$

$$v_* = v_0 + \frac{1 - F(v_*)}{f(v_*)} = v_0 + \frac{1 - \frac{v_* - \underline{v}}{\overline{v} - \underline{v}}}{\frac{1}{\overline{v} - \underline{v}}} = v_0 + \overline{v} - \underline{v} - v_* + \underline{v} \Rightarrow v_* = \frac{v_0 + \overline{v}}{2}$$

Thus, the critical valuation in an optimal auction will be

$$v_* = \frac{v_0 + \overline{v}}{2}$$

which is just *mid-way* between the seller valuation and the maximum possible bidder valuation. So, any of the standard auctions with a reserve price of $b_0 = \frac{v_0 + \overline{v}}{2}$ are optimal.

2.8.2. Uniform distribution of valuations - simple

We specialize the preceeding example further: $n = 2$, $v_0 = 0$, $\underline{v} = 0$, $\overline{v} = 1$, $F(v) = v$, $f(v) = 1$. This immediately leads to

$$v_* = \frac{1}{2}$$

Optimal auction: Any standard auction with reserve price $b_0 = v_* = \frac{1}{2}$. ER in optimal auction is given by

$$ER = \int_{\frac{1}{2}}^{1} MR(v) f^*(v) dv = 2 \int_{\frac{1}{2}}^{1} (2v - 1) v dv = \frac{5}{12}$$

Standard auction without reserve: ER in standard auction is given by

$$ER = 2 \int_{0}^{1} (2v - 1) v dv = \frac{1}{3}$$

We conclude that a reserve price (*set optimally*) increases expected seller revenue (= payoff since $v_0 = 0$) by approximately 25%!

Just to drive this home, let us also consider the bidding strategies in an FPSB auction with and without the reserve price.

Optimal FPSB auction: The bidding strategy is given as follows

$$b(v) = v - \frac{1}{v}\int_{\frac{1}{2}}^{v} x dx = v - \frac{1}{v}[\frac{1}{2}x^2]_{\frac{1}{2}}^{v} = \frac{v}{2} + \frac{1}{8v}, \text{ for } v \geq \frac{1}{2}$$

Standard FPSB auction: The bidding strategy is given as follows

$$b(v) = v - \frac{1}{v}\int_{0}^{v} x dx = v - \frac{1}{v}[\frac{1}{2}x^2]_{0}^{v} = \frac{v}{2}, \text{ for } v \geq 0$$

Thus, the active bidders bid more aggressively in the FPSB auction with the reserve price.

2.8.3. Examples from Riley & Samuelson

All these examples are set in the simple uniform case: $n = 2$, $v_0 = 0$, $\underline{v} = 0$, $\overline{v} = 1$, $F(v) = v$, $f(v) = 1$

Example 1: *Sad Loser Auction*

Rules
(i) Entry fee C (allows one bid)
(ii) High bidder wins, but *pays nothing*
(iii) Low bidder (if there is one) *pays his bid* (This is, indeed, a sad loser)

"*Intuition*" (wrong!!): No equilibrium since bidding "at" infinity.

This is wrong, and the auction does, in fact, belong to the family $\Re$!
Why? The expected payoff to an active bidder with value v_i who pretends to have value x is

$$\begin{aligned} u(x, v_i) &= v_i \times pr\{b(x) \text{ is the winning bid}\} - b(v) \times pr\{b(x) \text{ a losing bid}\} - C \\ &= v_i F(x) - b(x)(1 - F(x)) - C \end{aligned}$$

If $b(v)$ is a BNE strategy, then

$$P(v) = b(v)(1 - F(v)) + C = b(v)(1 - v) + C$$

must be the expected total payments for a bidder with valuation $v \geq v_*$. We already know that $v_* = \frac{1}{2}$.

What about $b(v_*)$? $b(v_*) = 0$, since there is always a loss if there is another active bidder, and the bid $b(v_*) = 0$ is inconsequential if there is no other active bidder. Hence, to implement v_*, the fee should be set such that

$$v_* F(v_*) - b(v_*)(1 - F(v_*)) - C = v_* F(v_*) - C = 0$$

or

$$C_* = v_* F(v_*) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$$

Also, we have

$$P(v) = vF(v) - \int_{v_*}^{v} F(x)dx = v^2 - \int_{\frac{1}{2}}^{v} xdx = v^2 - \frac{1}{2}v^2 + \frac{1}{8} = \frac{1}{2}v^2 + \frac{1}{8}$$

We equate the two expressions for $P(v)$ to get

$$b(v)(1 - v) + C_* = \frac{1}{2}v^2 + \frac{1}{8}$$

and we obtain the bidding function as

$$b(v) = \frac{\frac{1}{2}v^2 + \frac{1}{8} - C_*}{1 - v} = \frac{\frac{1}{2}v^2 + \frac{1}{8} - \frac{1}{4}}{1 - v} = \frac{v^2 - \frac{1}{4}}{2(1 - v)}$$

Now, since $b'(v) > 0$, we conclude that the auction belongs to the family $\Re$. Note that $b(v) \to \infty$ for $v \to 1$. But, The Sad Loser Auction is *optimal* when $C = \frac{1}{4}$.

Example 2: *Santa Claus Auction*

Rules

(i) A bidder who submits $b \geq v_*$ receives $S(b) = \int_{v_*}^{b} F(v)dv$ from the seller (like a "gift" from Santa Claus)

(ii) The high bidder wins and pays his bid, so that his net payment is $b - S(b)$

Claim: Optimal to bid $b = v$ (one's true value).
Proof. Suppose bidder 2 bids his value $b_2 = v_2$. Then, bidder 1's payoff from bidding $b_1 \geq v_*$ is

$$(v_1 - b_1) \times pr\{b_1 \text{ is the winning bid}\} + S(b_1) = (v_1 - b_1)F(b_1) + S(b_1)$$

Bidder 1 seeks to maximize this. First-order condition,

$$(v_1 - b_1)f(b_1) - F(b_1) + F(b_1) = 0$$

But this requires $b_1 = v_1$. ■

Given this, it is immediate that bids are increasing in v. $\rightarrow$ The auction belongs to the family $\Re$, and we know that $v_* = \frac{1}{2}$. To derive the expected revenue (= payoff) to the seller, we note that

$$f_{1,2}(x) = 2f(x)F(x) = 2x$$

and

$$S(b) = \int_{\frac{1}{2}}^{b} x dx = \frac{1}{2}b^2 - \frac{1}{8}$$

Now, expected seller revenue can be written as

$$\begin{aligned} ER &= \int_{\frac{1}{2}}^{1} v f_{1,2}(x) dv - 2\int_{\frac{1}{2}}^{1} S(v) dv \\ &= 2\int_{\frac{1}{2}}^{1} v^2 dv - 2\int_{\frac{1}{2}}^{1} [\tfrac{1}{2}v^2 - \tfrac{1}{8}] dv \\ &= \int_{\frac{1}{2}}^{1} v^2 dv + \tfrac{1}{4}\int_{\frac{1}{2}}^{1} dv \\ &= [\tfrac{1}{3}v^3 + \tfrac{1}{4}v]_{\frac{1}{2}}^{1} = \tfrac{5}{12} \end{aligned}$$

Recalling our previous results for this case, we conclude that the Santa Claus Auction is optimal.

Example 3: *Matching Auction*

Rules

(i) First, bidder 1 may bid any price $b_1 \geq v_*$

(ii) Then, if bidder 1 has made a bid, bidder 2 can *match* this bid, and he wins at price $b_2 = b_1$ if he does. If bidder 1 has made no bid, bidder 2 can obtain the object for a price of v_* if he wants

This mechanism is familiar from e.g. the housing market: Current occupier gets to match the bid of an outsider. This mechanism seems natural, but it violates the assumptions (non-discrimination and strict monotonicity) of the first theorem from Riley and Samuelson (1981), and the mechanism is not optimal. To see this, consider the strategies of the two bidders.

Bidder 2's strategy: Bidder 2 will match the bid of bidder 1 if and only if $v_2 \geq b_1$. If bidder 1 makes no bid, bidder 1 buys at the price $v_* = \frac{1}{2}$ if $v_2 \geq \frac{1}{2}$.

Bidder 1's strategy: Bidder 1 seeks to solve

$$\begin{aligned} & \max_{b_1}\{(v_1 - b_1) \times pr\{b_1 \text{ is the winning bid}\}\} \\ \sim \quad & \max_{b_1}\{(v_1 - b_1) \times pr\{b_1 \text{ will not be matched}\}\} \\ \sim \quad & \max_{b_1}\{(v_1 - b_1) \times pr\{v_2 < b_1\}\} \\ \sim \quad & \max_{b_1}\{(v_1 - b_1) b_1\} \end{aligned}$$

Take the derivative of the payoffs

$$\frac{d(v_1 - b_1) \cdot b_1}{db_1} = v_2 - 2b_1 \leq 0, \, \forall b_1 \geq \frac{1}{2}$$

It follows that the optimal bidding strategy of bidder 1 is

$$b_1(v_1) = \begin{cases} 0 & if \quad v_1 < \frac{1}{2} \\ \frac{1}{2} & if \quad v_1 \geq \frac{1}{2} \end{cases}$$

This is not a strictly increasing function of the valuation, and we conclude that the Matching Auction is not in the family $\Re$. Further, if $v_2 \in (\frac{1}{2}, v_1)$, then bidder 2 wins even though $v_1 > v_2$ ($\rightarrow$ *Inefficiency*).

What about the expected revenue to the seller? The actual revenue, R, can be written as

$$R = \begin{cases} 0 & if \ \max\{v_1, v_2\} < \frac{1}{2} \text{ (happens with probability } \frac{1}{4}) \\ \frac{1}{2} & otherwise \text{ (happens with probability } \frac{3}{4}) \end{cases}$$

We conclude that

$$ER = \frac{3}{8} < \frac{5}{12}$$

and it follows that the Matching Auction is *not* optimal.

2.8.4. All-pay format

Return to the general case with n bidders, and v_i distributed according to some $F(\cdot)$ on $[\underline{v}, \overline{v}]$.

- All-pay format - All bidders pay amounts tied to the vector of bids.
- Simple all-pay auction - every active bidder pays his bid $b(v)$, and the seller can set a reserve price.

Recall from the general case that a bidder's expected payment as function of his value is

$$P(v) = vF^{n-1}(v) - \int_{v_*}^{v} F^{n-1}(x)dx, \, \forall v \geq v_*$$

Further in the all-pay auction described, the expected payment is just one's bid, that is,

$$P(v) = b(v)$$

Hence, the bidding function is

$$b(v) = vF^{n-1}(v) - \int_{v_*}^{v} F^{n-1}(x)dx$$

Thus, for the critical valuation $v = v_*$, the bid must be $b(v_*) = v_* F^{n-1}(v_*)$. Then, a bidder with this particular valuation is indifferent between participating and not, since the expected payoff from participating is

$$v_* F^{n-1}(v_*) - b(v_*) = 0$$

We also know that

$$v_* = v_0 + \frac{1 - F(v_*)}{f(v_*)}$$

So, to implement v_*, the seller should set the reserve price b_0 as

$$b_0 = v_* F^{n-1}(v_*)$$

where v_* solves $v_* = v_0 + \frac{1-F(v_*)}{f(v_*)}$. Further, the bidding function $b(v)$ above is strictly increasing, and it follows that a simple all-pay auction with reserve $b_0 = v_* F^{n-1}(v_*)$ is optimal in the IIPV case.

Specialize example: Go to uniform case with n bidders and $v_0 = 0$. Then $v_* = \frac{1}{2}$, and we have

$$b_0 = v_* F^{n-1}(v_*) = v_*(v_*)^{n-1} = (v_*)^n$$

Hence,

$$b_0 = (\frac{1}{2})^n$$

In other words, in the *optimal* simple all-pay auction, the reserve price is decreasing in the number of bidders (unsurprisingly).

2.9. Bidder risk aversion - IIPV case

Based on Riley & Samuelson (Section 3). Bidders are identically *risk averse*, that is, they have the same utility functions. Drop Assumption A1, but retain the rest (notably, the seller is still risk neutral).

Rough summary of results

1. RET (for standard auctions) breaks down
2. OAB/SPSB auctions are *unaffected* by bidder risk aversion - same dominant strategies - same expected revenues

Corollary 2. *The seller can never do worse when bidders are risk averse rather than risk neutral.*

Surprising?? (see below) Can the seller do better??
Riley & Samuelson state the following result

Theorem 2.5. *Suppose IIPV and common bidder utility functions exhibiting risk aversion. Then*

(i) In OAB/SPSB auction, $b(v) = v$

(ii) In Dutch/FPSB auction, bidders bid more aggressively, the more risk averse they are

(iii) Dutch/FPSB auction generates higher ER than OAB/SPSB auction

Proof (sketch, see Riley & Samuelson for the details): (i) follows from (2) above, and (iii) follows from (ii). Hence, we only need to consider (ii). ■

Assumptions

- $u(x)$ - is the common utility function, which is strictly increasing and concave (risk aversion), and $u(0) = 0$.
- $b(v)$ - is the common bidding function.

Intuition (on bidder behavior in Dutch/FPSB auctions)

- If a bidder loses, he gets nothing
- If a bidder increases his bid a little, he gains a little *less* if he wins, *and* he *increases* the probability of winning. Thus, if he is risk averse he has an incentive to increase the bid above the risk neutral case.
- The *more* risk averse, the *more* a bidder will do to prevent ending up as a loser. Thus, he increases his bid as the degree of risk aversion increases.

This implies that the seller will prefer the standard Dutch/FPSB auction over the standard OAB/SPSB auction when bidders are risk averse. Standard Dutch and FPSB auctions generate a higher expected revenue - RET breaks down.

A final result from Riley & Samuelson is the following.

Theorem 2.6. *The optimal reserve price for the seller is decreasing in the degree of bidder risk aversion*

Proof: See Riley and Samuelson. ■

Intuition

The more risk averse the bidders are, the higher they bid. Hence,

- the need for the seller to induce aggressive bidding through a reserve price is less
- the seller has an incentive to lower b_0 to decrease the probability of missing favorable trades

Optimal auctions with risk averse bidders?

Requires a more refined analysis which we will skip here.

3. Common and Affiliated Values

We just want to scratch the surface. Generally, the uncertainty about bidder valuations does *not only* arise from different "tastes"/"technologies". In addition, it may follow from bidders having access to different pieces of information, leading to different *estimates* of the value of the item.

Drop A2 and retain the rest.

Dependent vs. independent signals.

Examples

Oil drilling - US OCS auctions - geological surveys
Airwaves - market analyses
Government bonds - market forecasts

3.1. (Pure) Common Value Setting

Special case in which valuations are *perfectly, positively* correlated → The item has a unique true/objective value to bidders (a *common* value) → But bidders have access to different pieces of information about this common value ⇒ Different *estimates* of the true value.

Some notation:

v - true (common) value
x_i - bidder i's imperfect information/signal about v

Assumptions:

a) $x_i > x_j \Rightarrow E(v \mid x_i) > E(v \mid x_j)$. The estimate of v is increasing in the observation x.

b) $H(x \mid v)$ is the common distribution of the x_i's given the true value v.

c) $E(x \mid v) = v$. Thus, the expected value of the signal coincides with the true value.

The Winner's Curse

Suppose that bidder i *naively* bids $b_i = b(E(v \mid x_i))$. So, he bids purely on the basis of a simple extimate of v given his own signal x_i. This would give rise to

> The Winner's Curse: A naive bidder will typically observe that he does very poorly when he wins.

Why? Let $X_{[1]}$ be the largest signal observed among the n bidders. Then

$$E(x \mid v) = v \Rightarrow E(X_{[1]} \mid v) > v$$

This just reflects that the expected *maximal* draw from a given distribution exceeds the *average* draw, which should be near the true value,v, in a large sample. $\Rightarrow$

> If bidder i wins and has made a bid "close" to the signal $x_i = X_{[1]}$ (allowing for a standard profit margin), then he will typically have bid too much (and lost money). This is The Winner's Curse.

Formally

$$E(v \mid x_i) > E(v \mid x_i, x_i > x_j, \forall j \neq i)$$

Hence, being a winner in this case conveys bad news $\rightarrow$ Everybody else have observed lower signals and made lower bids $\rightarrow$ This is a strong indication that x_i overestimates v!!

Sophisticated bidding (suppose we are in a FPSB auction)

A sophisticated bidder should *shade* his bid below his estimate of v for *two* reasons:
(1) To allow a profit margin
(2) To overcome The Winner's Curse

In other words: A *rational* bidder should assume that he has the highest signal, and then bid optimally according to this assumption $\rightarrow$ Bids are lower than under *naive* bidding, since the expectation of v is lower!

3.2. Affiliated Values Setting

Affiliation ("correlation"): Valuations are said to be *affiliated* if the fact that one bidder has a high value estimate makes it likely (probabilistically) that other bidders also have high value estimates. $\Rightarrow$

The OAB format differs significantly from SPSB, FPSB and Dutch auction formats

Why? The bidding process in an OAB auction *reveals information* to the bidders. Remaining bidders observe when others drop out.

- In IIPV (the bench-mark) model, this does not change bidder behavior and ER
- When valuations are affiliated, the observed behavior in the OAB auction partly reveals (makes public) each bidder's private information, thus alleviating the winner's curse phenomenon (Milgrom & Weber (1982), see Klemperer (1999))

This gives rises to the following results which we state without proof (see Milgrom & Weber (1982)).

Theorem 3.1. *When valuations are affiliated, the OAB auction yields higher ER than SPSB, FPSB and Dutch auctions.*

Theorem 3.2. *From the point of view of the seller, OAB $\succ$ SPSB $\succ$ Dutch/FPSB.*

Affiliation and optimal auctions

Analytically difficult, and we leave this for some other time!

4. Asymmetric Bidders

Often, we are likely to be in a situation where the bidders are not symmetric (ex ante identical). Drop A3 and A4, but retain the rest. Dropping A4 (non-discrimination) comes natural, since we may have some information on particular bidders, which might be used in the design of the auction.

What we have in mind is the following: The auction designer is able to distinguish between different kinds of potential bidders in an auction. Different kinds of potential bidders have valuations drawn from different distributions. How might he use this in the design of the auction?

Tie in with the price discriminating monopolist:

The monopolist would ideally want to price discriminate perfectly (1st degree). But he does not have the requisite information, and he might not be able to prevent resale.

So, in practice, he might try a less ambitious type of price discrimination

Market segmentation (3rd degree)
Sorting/self-selection (2nd degree)

depending on the exact nature of the information he has.

Information of the auctioneer:

(1) On the actual kind of particular bidders
Auctioneer may know whether i is a dealer or private collector, an incumbent or a potential entrant, a domestic firm or a foreign firm, etc., etc.

(2) On the kinds that a bidder may be, but not who is who
This looks a lot like what we have considered so far (but the case is somewhat richer)

Let us focus on the first case which is similar to the case of the 3rd degree price discriminating monopolist (who can prevent resale).

Examples:

(1) European UMTS: Incumbents (from 2G) versus new entrants. In the UK case, the incumbents (BT/CellNet, Vodafone, Orange & One2One) were not allowed to bid at all on the A License, but only on licenses B through E. As it turned out, the incumbents won B, C, D and E, respectively, while an entrant won A.

(2) US PCS: Dedicated frequencies. In certain areas, certain frequencies were labelled as dedicated. This meant that particular groups (e.g. ethnic minorities) were given an advantage, in the sense that they only had to bid within, say, 10% of an ordinary commercial bidder to win.

(3) Generic case: It might be possible to distinguish, in advance, between commercial bidders (experts) and private bidders (non-experts). The auction rules may treat these groups differently.

4.1. Private value differences

Optimal (revenue maximizing) auction allocates object(s) to the bidder(s) with the highest marginal revenue(s) rather than to those with the highest value(s) (Klemperer (1999, 7.1)).

Assume that “strong” bidders have individual demands that are shifted (horizontally) left compared to those of “weak” bidders.

Then, a buyer/bidder on a given demand curve has a higher marginal revenue than any buyer/bidder with the same value on a higher demand curve.

Thus, an optimal auction from the point of view of the seller would discriminate in favour of bidders with valuations drawn from lower distributions (the weak bidders).

What does this imply for the comparison of standard auctions with asymmetric bidders?

Difficult to answer generally, but Maskin and Riley (2000) give some results.

In a FPSB auction a bidder with valuation drawn from a low distribution bids more aggressively (closer to value) than bidder with valuation drawn from high distribution. Let a bidder consider raising his bid by $\triangle b$ in order to raise the probability of winning by $\triangle p$. The first-order condition of a bidder with value v can roughly be written as

$$(v-b)\triangle p - p\triangle b = 0$$

or

$$(v-b)\triangle p = p\triangle b$$

Since the probability of winning is increasing in v, it follows that a bidder with a relatively low valuation will bid closer to his value ($v-b$ is small) than a bidder with a higher valuation. So, the FPSB auction already "discriminates" in favor of the weak bidder as opposed to OAB and SPSB auctions, where everybody basically bid their valuations (IPV case).

So, in IPV wihtout symmetry it is conceivable that FPSB auction may be more profitable (at the possible expense of social efficiency) than the OAB and SPSB auctions. This may or may not be so. Klemperer quotes Maskin and Riley:

> "Roughly speaking, the sealed-bid [FPSB] auction generates more revenue than the open [OAB/SPSB] auction when bidders have distributions with the same shape (but different supports), whereas the open auction dominates when, across bidders, distributions have different shapes but approximately the same support."

Strong bidders tend to prefer OAB/SPSB auctions whereas weak bidders tend to prefer FPSB auction. Important when attracting bidders is important! E.g. UMTS.

Maskin & Riley (2000) example:

Two risk-neutral, potential buyers with independently distributed private values.
Weak bidder - $v_w \sim unif[0,1]$
Strong bidder - $v_s \sim unif[2,3]$
Compare expected revenues to the seller in FPSB and SPSB auctions.

FPSB

Assume that weak bidder bids his value, $b(v_w) = v_w$ (we check below that this is part of an equilibrium). What is the strong bidder's best response?

If she bids $b \in [0,1]$, her probability of winning is $F_w(b_w^{-1}(b)) = b$. Thus, her maximization problem is

$$\max_{b\in[0,1]} b(v_s - b)$$

The derivative w.r.t. b is $v_s - 2b \geq 0$ on $[0, 1]$ for $v_s \geq 2$ (as assumed). Hence, the strong bidder's best response is to bid $b_s(v_s) = 1$ for all $v_s \in [2, 3]$. Now, when the strong bidder bids $b_s(v_s) = 1$, she wins for sure, and it is a best response for the weak bidder to bid $b(v_w) = v_w$ (as assumed at the outset). Hence, we have derived a BNE where the strong bidder bids 1 and wins for sure, while the weak bidder bids his valuation and never wins.

$$\Rightarrow ER_{FP} = R = 1$$

Maskin & Riley have shown that the equilibrium is essentially unique (but we need not concern ourselves with this here).

SPSB

Optimal to bid one's value (by standard arguments), which implies that the strong bidder always wins and pays the bid of the weak bidder.

$$\Rightarrow ER_{SP} = E(v_w) = \frac{1}{2}$$

RET obviously breaks down since

$$ER_{FP} = 1 > ER_{SP} = \frac{1}{2}$$

Of course, the example is somewhat contrived, but Maskin & Riley have more general results along these lines, and these are very important for practical auction design.

4.2. Asymmetry and almost-common values

See Klemperer (1999, 7.2)

Almost-common values: Small private value component and one bidder has a small advantage (slightly higher private value).

May translate into huge competitive advantage in certain auction formats.

OAB vs. FPSB auctions

Examples

2G incumbents in 3G auctions

Upstream ownership stakes (media industry)

OAB

The player with the small advantage in the almost-common value case will bid slightly more aggressively → This strengthens the Winner's Curse of the slightly disadvantaged bidders (winning against a more aggressive bidder conveys even worse news - overestimation by even more) → Slightly disadvantaged bidders will bid more defensively → Winner's curse of advantaged bidder is reduced even more → Advantaged bidder will bid more aggressively still → And so on!

- Small toe-holds may translate into significant competitive advantage in OAB format.
- Winner may often have lower signal/MR, and OAB may not be very profitable from the point of view of the seller.

FPSB

- Continuous at the symmetric limit. Small private value differences translate into small changes in strategies and auction outcome.
- Bidder with highest signal/MR continues to win with very high likelihood
- So, in the near symmetric case, the FPSB auction continues to be (close to) revenue maximizing.

Conclusion

- OAB and FPSB auctions may be very different from the perspective of expected revenue in the almost-common values case
- Bidding costs may imply that even a slightly disadvantaged bidder may stay out of the OAB auction
- Serious concern in UMTS auctions

Toe–hold example

UK Premier League Soccer, ownership and TV revenues

BSkyB takeover of Manchester United blocked, but small stakes may do the trick

BSkyB stakes in Manchester United, Manchester City, Chelsea, Leeds, Sunderland

NTL stakes in Aston Villa, Leicester, Middlesborough, Newcastle

Response of The Premier League (association) - Change from OAB auction to format including sealed-bid stage

5. Collusion

Collusion may be explicit (bidding ring/cartel) or tacit. How do the standard auctions fare?

(I) Cartel/Bidding Ring

Suppose all bidders have joined and have agreed how the spoils should be divided/who should win.

SPSB auction

Let the designated bidder bid infinitely high (or, at least, very high) and the rest bid zero. Then the designated winner wins and pays nothing.

No incentives to cheat on the collusive agreement:

- The designated winner has nothing to gain by bidding higher or lower.
- No designated loser has anything to gain from bidding higher. To change the allocation he has to beat the designated winner, which is not worthwhile (by assumption).

FPSB auction

The winner pays his bid. So, to replicate the outcome of the SPSB case, the cartel members have to agree that the designated winner should bid zero and no one else should bid (or, the designated winner should bid a very small positive amount and all the designated losers should bid zero).

Strong incentives of losers to cheat on this agreement:

- A designated loser just has to beat the designated winner by a small amount to win, which leaves a lot of surplus to the cheating bidder (given that the designated winner was supposed to win with a bid at or close to zero).

Conclusion

Seems easier for a bidding ring to sustain collusion in a SPSB auction than in a FPSB auction!

Cartel enforcement:
Pre-auction or post-auction knock-outs
Weak or strong cartel (side-payments)
All inclusive vs. "dominant" group

(II) Tacit collusion

i. One shot case.

Suppose for illustration that the bidders know each others valuations (but the seller does not). Then, roughly, in an OAB the following strategies form an equilibrium: All but the highest valuation bidder drops out at zero. If not, all those who stayed at zero (plus a bit) stay in until the price reaches their value.

So, the high valuation bidder wins anyway, and it makes no sense for anyone but the high valuation bidder to stay in beyond zero. Hence, the high valuation bidder wins, but pays nothing.

ii. Repeated auctions.

Looks more like standard IO setting (repeated oligopoly), where strategies in subsequent periods can be contingent on what happened earlier, and standard results apply.

Example

Klemperer (2003b) on Ford UK online auctions of new cars
Dealers bidding against each other

- Traditional dealer sales - Resemble FPSB auction
- Online auction - Resemples OAB auction (in fact, it is a descending bid auction, but open)

Online auction may be more collusion-prone than traditional dealer sales - higher prices might be expected. Against this there is a decrease in search and transactions costs, which might pull in the other direction.

More general comment on online auctions in comparison to traditional modes of transaction. Important question: Who is organizing the auction/ exchange?

Remedies by seller?

Reserve price (secret?)
Lot size, lumpiness of orders (as in standard oligopoly setting)
Non-disclosure (winning bids, losing bids, identities of winners or bidders)
Discrimination or randomization of winner (if close bids)
Rules on communication

6. Multiple Units

Theory generally a lot less well-developed for the multi-unit case than for the single-unit case. Exception is case where each bidder only desires one unit (see the extension of the RET to this case above). The latter relevant for the UMTS auctions (each bidder may only be allowed to win one of several pre-specified spectrum packages - but this a decision the seller has to make). We shall just make a few comments on the multi-unit case where bidders may have general demand functions.

Much interest in recent years spurred by:

- Treasury auctions
- Radio-spectrum auctions
- Share auctions (tenders)
- Electricity exchanges

Assume that fixed quantity is auctioned:
This could be fixed supply of e.g. shares in a firm or radio frequencies
This could be a fixed demand for e.g. electricity
We refer also to the comments already made in the Introduction.

Issues
Simultaneous vs. sequential auctions
Homogenous vs. heterogenous goods
Complementarities between goods
Common value components
Externalities between bidders

6.1. Simultaneous auctions

Think of this in terms of supply function bidding for a fixed demand (e.g. electricity). Individual supply functions are added to clear demand.

Compare uniform-price auction and discriminatory auction
Uniform-price auction - Vickrey flavor
Price is where market clears (highest winning bid (ask) determines price)

Discriminatory auction - First-price flavor
All winners pay their bids (asks) for each unit

Uniform-price auction have BNE that look very collusive

Bidders can implicitly agree to divide the market by bidding very aggressively for quantities smaller than the agreed shares of demand. Heuristically, the individual supply functions very steep around the agreed shares. This makes it very expensive for a potential deviating bidder to capture more quantity by undercutting the rivals. So, bidders have no incentive to undercut, and the collusive outcome can be sustained.

Remedies?

(i) Discriminatory auction.

When winners have to pay their bids (asks) for all units won, it is very expensive to bid steep supply functions, and the equilibrium bid functions therefore tend to be flatter and the outcome more competitive to the benefit of the buyer.

(ii) Introduce randomness in the buyer's demand and in the supplies of non-competitive sellers (bidders).

This makes attempted collusion uncertain and potentially expensive and equilibria look more competitive.

Comment on this in light of electricity auctions (often only a few very large suppliers, and the buyer therefore has to think seriously about possible collusion).

6.2. Sequential auctions

Special case:

- No bidder wants more than one unit
- Units are homogenous
- Standard independence and symmetry assumptions

Then RET holds whether item sold simultaneously or sequentially - Flat price path!

But what if these assumptions are violated? Literature relatively sparse
Affiliation/Common value elements
Price may drift up in sequential auctions (information released - informational linkage)
So, what about the “Afternoon Effect”? (evidence??, wine??)
Risk aversion may drive up early prices?
Complementarities/Synergies/IRS may explain this?

6.3. Further remarks

Complementarities and aggregation
Combinatorial bids?
Collusion
Simultaneous vs. sequential auctions - open vs. sealed
Post-auction market structure “downstream”
Double-auctions - strategizing on both sides of the market

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Fig. 1

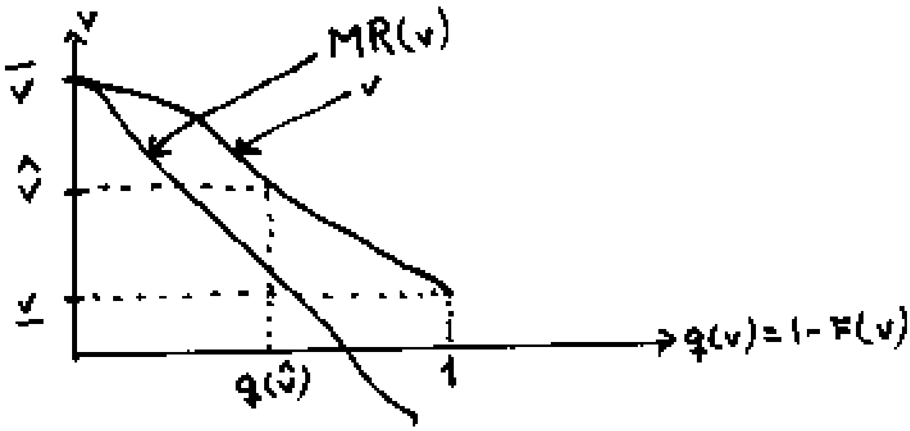

Fig. 2

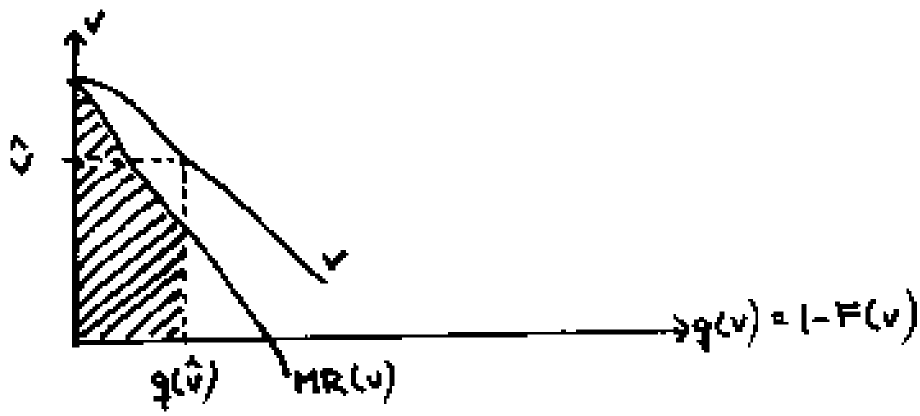

Fig. 3

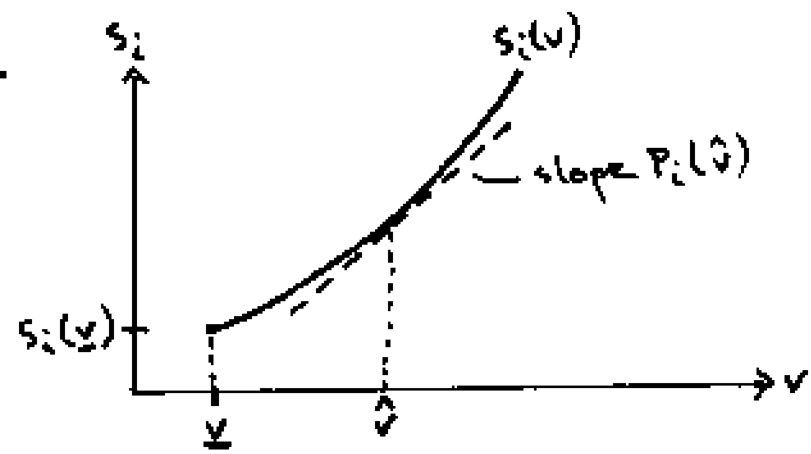

Fig. 4