Buy-Out Prices in Auctions: Seller Competition and Multi-Unit Demands*

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Abstract

Online auction sites often enable sellers to add a *buy-out* price. In one-shot auctions, this has been motivated by appeal to impatience or risk aversion. We offer additional justification in a dynamic model, by showing that an early seller has an incentive to use a buy-out price, if a similar product is offered later by another seller, *and* bidders desire multiple objects. Revenue in the first auction increases, but revenue in the second auction decreases, as does the sum of revenues. The buy-out price causes the auction sequence to become inefficient, since the first item may be awarded to a bidder who should have received none.

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1 Introduction

The proliferation of the *Internet* and online deal engines has brought a renewed focus on the details of auction design and their implications for bidder behavior and seller revenues. To name a few features which have drawn attention, we can list the relationship between closing rules and bidder sniping, open vs. secret reserve prices, jump bidding, signaling and default. In this paper we study the role of *buy-outs* in auctions. Roughly, the introduction of a buy-out option in an auction implies that the seller stipulates *ex ante* a price at which he is willing to end the auction immediately, if someone shows a willingness to pay this price.

In the economics literature, the presence of buy-out prices in *online* auctions[1] has thus far been explained by focusing on a *single* auction and assuming that potential buyers and/or sellers exhibit either *risk aversion* or *impatience* (see Budish and Takeyama (2001), Mathews (2003, 2004), Mathews and Katzman (2006), Reynolds and Wooders (2006) and Hidvégi, Wang and Whinston (2006)). In this paper we take a somewhat broader view of auction markets, realizing, in particular, that buyers and sellers alike are aware of the fact that new products will be offered on the market in the future. This will tend to depress revenue in today's auctions, as buyers know that close substitutes will be offered tomorrow. In this *dynamic* environment we will show that there is good reason for an *early* seller to introduce a buy-out price, even if agents are patient and risk neutral.[2]

Buy-out prices or maximum prices in online auctions were noted by Lucking-Reiley (2000) in his empirical overview of auction activities on the *Internet.* Since (sell) auctions are ostensibly staged to elicit high prices in situations where markets are thin and sellers are short on information about the willingness-to-pay of potential buyers, such buy-out prices may appear surprising. In fact, Lucking-Reiley explicitly posed this as a challenge to theorists. In addition, he quoted evidence to suggest that the *exercise* of posted buy-out options is not uncommon in online auctions. Hence, buy-out prices do more than just attract attention.

Reynolds and Wooders (2006) provide some additional information on the *frequency of buy-out prices* in *Yahoo!* and *eBay* auctions, though, *not* on the frequency with which the option was *exercised* by some bidder. The categories sampled on March 27, 2002, were automobiles, clothing, DVD players, VCR's, digital cameras and TV sets. A total of 1248 auctioned items were sampled from *Yahoo!*, of which 842 had a buy-out price posted by the seller (roughly, 66%). In similar fashion, 31142 auctioned items were sampled from *eBay*, of which 12480 had a buy-out price posted by the seller (roughly, 40%). There is some variation across the categories of goods sampled, but the frequency of buy-out prices never drops below 25% in the sample. Hence, in these categories, at least, the appearance of buy-out prices is very frequent.

For *eBay*, Mathews (2004) presents some numbers on the *frequency with which buy-out options are exercised* when offered. For two categories of games (racing and sports) for *Sony PS2*, Mathews reports that on January 29 - 30, 2001, 210 items were on offer. A buy-out option was available on 124 items (59%), and it was exercised 34 times (27% of the times it was offered). So, at least in these categories, the exercise frequency is high.

Formally, we analyze *eBay*'s version of a buy-out price, termed the *Buy It Now* price (introduced in January 2001). Here is how the *Buy It Now* price roughly works from the seller's viewpoint:[3] "If a buyer is willing to meet your *Buy It Now* price before the first bid comes in, your item sells instantly and your auction ends. Or, if a bid comes in first, the *Buy It Now* option disappears. Then your auction proceeds

[1] Alternatively, this is often referred to as *buy prices* or *maximum prices.* In offline settings, this phenomenon also has a certain affinity with "$xx or best offer", where it is, presumably, implicit that, if someone makes an offer of $xx, then the trade is finalized immediately, while if someone makes a lower offer initially, then the seller will wait a while to see if a better offer comes along. Also, a buy-out price has a certain similarity with a massive *jump bid* intended to end an auction quickly.

[2] Throughout this paper potential buyers bid *non-cooperatively.* In future work we hope to return to the use of buy-out prices in auctions where sellers try to respond to possible *bidder collusion.*

[3] For more details on the *eBay* version and other versions of a buy-out price, see the references above.

normally." Hence, in *eBay* auctions, the buy-out price is temporary.

Throughout this paper we assume that potential buyers or bidders have *multi-unit demands*, with diminishing marginal utility. With two objects for sale, at least two bidders and absent any buy-out option, it has been shown by Black and de Meza (1992) that expected auction revenue will increase over time and that the auction outcome is efficient under these assumptions. In particular, in a sequence of standard second-price or English auctions, the seller offering his good today will not earn as much as a competing seller offering a similar good tomorrow, that is, prices are increasing in expectation. In fact, Black and de Meza (1992) were mainly interested in what has been referred to as *The Declining Price Anomaly*. Therefore, they went on to consider an option of the following kind: the winner of the first item is given the option of buying the second item at the same price. This type of option is observed in certain multi-unit auctions, and it is enough to lead to a declining price path.

However, for the case with *two competing sellers*, we show that the first (i.e., the *early*) seller can always increase his revenue by introducing a buy-out price. The revenue to the second seller is adversely affected, as is overall revenue. An optimally chosen buy-out price in the first auction also introduces *inefficiency*, in the sense that *a bidder who should have won no object wins one*. Our analysis is partial in the following sense. We consider a sequence of two second-price (or English) auctions, allowing the first seller the possibility of introducing a buy-out price without giving the second seller the opportunity to respond in kind.[4] Thus, we essentially show that an auction market without buy-out prices is unstable, in the sense that current sellers will try to persuade the auction site to (at least temporarily) allow buy-out prices.[5]

At *eBay* and other auction sites, a seller can also introduce a reserve price, which we ignore in this paper. As is well known, reserve prices are generally useful because they allow sellers to ration or withhold items by effectively excluding potential buyers with low valuations. Reserve prices, thus, affect the probability with which objects are sold. As suggested in the previous paragraph, the effect of buy-out prices is fundamentally different. In particular, buy-out prices do not influence the probability that objects are sold, but they may change the identity of the winners. It follows that a buy-out price is not a substitute for a reserve price, and that it may have a role to play, even when a reserve price is present.

The rest of the paper is organized as follows. In Section 2 we set up a simple model and present the results for the bench-mark case where a sequence of two, standard second-price auctions is staged. Then, Section 3 shows that the first of two sellers can improve his lot by offering a buy-out price and presents results on the path of revenues, the properties of an optimal buy-out price and the overall efficiency of the string of auctions. In Section 4 we comment further on the relationship between the buy-out price, total revenue and efficiency. In addition, this section remarks on the robustness of our main result to changes in the auction format (second-price vs. English auction) and the nature of the buy-out option (temporary vs. permanent). To further illustrate the results, Section 5 briefly outlines an example with uniformly distributed valuations. Section 6 contains a few concluding remarks, while a selection of proofs

[4] In one-shot auctions with symmetric and risk neutral bidders, a buy-out price decreases expected revenue. However, bidders are not symmetric in the last auction because one of them won the first auction. In addition, bidding in the first auction may have revealed some information about the losers. It is currently not known whether a buy-out price is profitable in auctions with asymmetric bidders. However, Kirkegaard and Overgaard (2007) show that a buy-out price that is only available to one of the bidders is unprofitable in a second price auction (though it may be profitable in a first price auction). If bidders arrive to the second auction over time, the (temporary) buy-out price is effectively available only to the bidder who arrived first. In this case, Kirkegaard and Overgaard's (2007) result suggests that the second seller would not offer a buy-out price even if it was possible (however, Kirkegaard and Overgaard (2007) assume the distribution functions are not degenerate, but this assumption may not be satisfied in the second auction, given information is revealed in the first auction). Though we model the first auction as if all bidders arrive at the same time, the argument below at the end of Section 4.3 implies that this assumption is unimportant.

[5] In Kirkegaard and Overgaard (2003) we also considered the usefulness of buy-out prices for *one seller* intending to sell two objects. Such a seller can benefit from using a buy-out price in the *second* auction. This gives rise to a different type of ineffICENcy in the sequence of auctions. In particular, it is possible that *a bidder wins two items when he would only have won one* in an efficient auction.

is in the Appendix.[6]

2 Model and Bench-Mark

In this section we first set up the model and then derive results for the bench-mark case where a sequence of two second-price auctions is staged.

We assume that two identical objects are offered for sale sequentially, and that there are n potential buyers on the market. Each buyer i, $i = 1, 2, ..., n$, is characterized by a type, v_i, drawn from a continuously differentiable distribution function, $F(v_i)$, without mass points. The associated density is referred to as $f(v_i) = \frac{dF(v_i)}{dv_i}$, whereas further regularity assumptions on F will be imposed below (as the need arises). We assume that $v_i \in [0, \overline{v}]$. The value to bidder i of the first unit purchased is v_i, while the value of the second unit is kv_i, $0 < k < 1$. Hence, each bidder desires both units, but individual demands are downward sloping.

Below, we shall occasionally take the perspective of a particular bidder, i, and label his rivals j, $j = 1, 2, ..., n-1$. Now, i's competitors have random valuations of the first item denoted y_j which we order as $y_1 \geq y_2 \geq .... \geq y_{n-1}$. This allows us to refer to bidder j as bidders i's jth-strongest rival. When appealing to the order-statistics, we shall generally refer to $F_{m,n}(x)$ as the distribution function of the m'th-highest of n draws, with associated density $f_{m,n}(x) = \frac{dF_{m,n}(x)}{dx}$.

Throughout, we assume that two different sellers each own one object initially. The two objects are offered sequentially, and we allow the first seller to stipulate a buy-out price of the *eBay*-variety (*Buy It Now*). Thus, in the general case we consider the following augmented game:

1 Seller 1 announces a buy-out price, B. At this stage bidders can submit a bid of B or refrain from bidding. The object is sold at the price B if at least one bidder bids B. If several bidders bid B, one bidder is picked at random to win. If no one bids B, a normal second-price auction is staged. The price can exceed B in this event.

2 Seller 2 auctions off the second item, using a second-price auction.

In stage *one* of this game, the bidders first have to decide whether to take the buy-out price B or leave it. If one or more bidders take the buy-out price, the first auction ends, and the winner pays B. If no one takes the buy-out price, then the first stage continues to a standard second-price auction. Stage *two* simply consists of a standard second-price auction.

First, though, we summarize the results of the bench-mark case, where no buy-out price can be stipulated by the first seller (or, that it is set so high as to be irrelevant for the play of the game).

2.1 The bench-mark: Two straight second-price auctions

To keep the analysis simple, we ignore the use of reserve prices in the following.[7] In this setting, Black and de Meza (1992) were the first[8] to solve for equilibrium strategies in a sequence of two second-price, sealed-bid auctions, under more general assumptions than those considered here.[9] Applied to our set of assumptions, they find the following.

[6] Full details of the proofs as well as the example outlined in Section 5 are in the Web Appendix (see **www.tobeadded.com**).

[7] In fact, we are not aware of *any* papers on *sequential* auctions which incorporate reserve prices, even in the absence of buy-out prices.

[8] See also Katzman (1999).

[9] Black and de Meza explicitly consider sealed-bid auctions, while they also have an informal discussion of English auctions. Février, Roos and Visser (2005), on the other hand, focus on English auctions. See Section 4 below for some further comments.

Proposition 1 (Black and de Meza (1992)) *When there are n a priori symmetric bidders in the game, the unique symmetric equilibrium is for agent i to bid $E(\max\{ky_1, y_2\} \mid y_1 = v_i)$ in stage one, and to bid v_i in stage two if stage one was lost, and kv_i otherwise. The equilibrium outcome is efficient, and expected revenues are increasing from the first to the second auction.*

Thus, in the last round, a bidder simply bids his valuation of the remaining object, which depends on whether the bidder won or lost the first object. This is a weakly dominant strategy, as long as there are no costs associated with participating in the auction. Adding participation costs to the model may alter bidding behavior significantly. If bidders continue to use pure strategies in the first stage, meaning that the price in stage one reveals the valuation of the runner-up, the winner of stage one will know whether he will win or lose the second stage, should he decide to participate. In the latter case, it would be optimal for him to refrain from bidding, if there was even a miniscule participation cost (unless the runner-up is discouraged from participating in stage two given his more limited knowledge, specifically that he did not win stage one). While we assume in this paper that there are no participation costs, we refer to Tan and Yilankaya (2006) for a study of one-shot second-price, sealed-bid auctions with participation costs and potentially asymmetric bidders. Interestingly, Celik and Yilankaya (2006) show that the *optimal* auction may be significantly impacted by the presence of participation costs. In related work, Kirkegaard (2005) shows that a reserve price is more profitable than a participation fee in second-price, sealed-bid auctions with asymmetric bidders.

In the first round, each bidder essentially bids what he expects it to take to win the second item if he just loses the first. This, however, is the expectation of the maximum of k times his valuation for the first item won and the valuation of the first unit won by his second-strongest rival.

To see more clearly what is going on here, we first note that in case of *symmetric*, *increasing* bidding strategies, the fine details of any bidder's bid function are only consequential if there happens to be a rival bidder who has a valuation very close to that of the bidder in question. Hence, in equilibrium a bidder's strategy is pinned down by an *indifference relation*: the bidder should be indifferent between winning and losing, if his strongest rival is identical to himself. With this in mind, consider the *first* round. We note that if i is the "top dog" and there is someone like i in the pack of rivals, then they each win one item in equilibrium.[10] Hence, optimal bidding by i in the first round is derived from an indifference between, on the one hand, just winning the first and losing the second item and, on the other hand, just losing the first and winning the second item. Formally, this can be written as

$$[v_i - \underbrace{\qquad b^1(v_i) \qquad}_{b^1(y_1) \text{ with } y_1 = v_i}] + 0 \ = \ 0 + [v_i - E(\max\{ky_1, y_2\} \mid y_1 = v_i)]$$

Thus, in the first auction, bidder i should bid what he expects to have to pay to win the second, if he just loses the first. Rearranging, we have

$$b^1(v_i) = E(\max\{ky_1, y_2\} \mid y_1 = v_i) = E(\max\{kv_i, y_2\} \mid y_1 = v_i)$$

We conclude that bidder i should bid the expectation of the *maximum* of k times his strongest rival's valuation of the first item and his second-strongest rival's valuation of the first item *predicated on the strongest rival being identical to himself.* Note that with two bidders, y_2 is zero by construction, and the optimal bid of i in the first auction reduces to $b^1(v_i) = E(\max\{ky_1, 0\} \mid y_1 = v_i) = kv_i$. Hence, in the first round a bidder simply bids as he would in a second round after winning the first.

[10] When strategies are symmetric and increasing, the first auction is won if the toughest rival has a lower valuation, and lost if the toughest rival has a higher valuation. If the toughest rival has the same valuation as the agent himself, there is a tie, and the winner of the first auction is determined by chance. We argue that the agent must be indifferent between winning and losing the first auction in this case. Assume, to the contrary, that the agent prefers to win (lose) against an identical, strongest rival. Then, the agent should bid more (less) aggressively at the outset to win (lose) with probability one (rather than one half). This implies that the original strategies are not in equilibrium, unless the indifference condition is satisfied.

Essentially, when bidders have *multi-unit demands* ($k > 0$), the reasons why the expected revenue in the second auction is higher than in the first are as follows. Since auctions are both "second price", their prices (hence, revenues) are determined by the runners-up, that is, the bidders with the second-highest marginal valuations. Furthermore, any bidder bases his bid in the first auction on the assumption that his strongest rival has the same valuation. Thus, if the runner-up and the winner of the first auction do indeed have the same valuations, revenues will be constant, which is just another way of stating the (*ex ante*) indifference condition. Of course, the probability that the realized valuations of two bidders coincide is zero. Hence, whenever the realized valuations of the two strongest bidders differ, the actual runner-up in the first auction will effectively (*ex post*) have "underestimated" the intensity of competition and, consequently, the bid it takes to win the second auction. It follows that expected prices (revenues) are increasing. In contrast, with *unit-demands* ($k = 0$), the fact that the runner-up has a strictly lower valuation than the winner is irrelevant, since the winner of the first auction does not compete in the second auction. In this case, expected prices (revenues) are constant.

The next result summarizes the first-round bidding and the associated expected revenues. To facilitate the exposition in the following, we let $m(v) = \min\{\frac{v}{k}, \overline{v}\}$. Thus, if the runner-up in the first auction has type v, he will win the second auction, if the highest rival type (that belonging to the winner of the first auction) is between v and $m(v)$, since in that case the winner of stage one will bid at most $km(v) \leq v$. However, if the highest rival type is between $m(v)$ and $\overline{v}$, the bidder will be the runner-up in both auctions. For subsequent comparison with our main findings, we state the following simple results.

Lemma 1 *The optimal bidding strategy in the first auction is*

$$b^1(v) = \frac{kvF_{1,n-2}(kv) + \int_{kv}^{v} xf_{1,n-2}(x)dx}{F_{1,n-2}(v)} \tag{1}$$

Thus, in two straight second-price auctions with n bidders, the expected revenues in the first and second auctions are, respectively,

$$\begin{aligned} ER_1^{SSP} &= \int_0^{\overline{v}} kv\frac{F^{n-2}(kv)}{F^{n-2}(v)} f_{2,n}(v)dv \\ &\quad + \int_0^{\overline{v}} v \cdot \frac{(1-F(v))^2 - (1-F(m(v)))^2}{(1-F(v))^2} \cdot f_{3,n}(v)dv \end{aligned} \tag{2}$$

and

$$\begin{aligned} ER_2^{SSP} &= \int_0^{\overline{v}} v \cdot \frac{1-F(m(v))}{1-F(v)} \cdot f_{2,n}(v)dv \\ &\quad + \int_0^{\overline{v}} kv \cdot \frac{F(v)-F(kv)}{1-F(v)} \cdot \frac{F^{n-2}(kv)}{F^{n-2}(v)} f_{2,n}(v)dv \\ &\quad + \int_0^{\overline{v}} v \cdot \frac{(F(m(v))-F(v))^2}{(1-F(v))^2} \cdot f_{3,n}(v)dv \end{aligned} \tag{3}$$

Proof. See the Web Appendix. ■

We note that $ER_1^{SSP} \to \int_0^{\overline{v}} vf_{3,n}(v)dv = E(v_{[3]})$ and $ER_2^{SSP} \to \int_0^{\overline{v}} vf_{3,n}(v)dv = E(v_{[3]})$ as $k \to 0$, where $E(v_{[3]})$ is the expectation of the third-highest of n independent random draws from $F(v)$. This, however, is just a special version of Weber's (1983) result that a sequence of second-price (or English) auctions where bidders have *unit demands* yields the same expected revenue to all sellers. Notice further that with only *two* bidders and two items for sale, the equilibrium revenue is zero to both sellers. It is impossible to extract rent from buyers when there is no excess demand, recalling our assumption of no reserve prices.[11]

Similarly, we note that $ER_1^{SSP} \to \int_0^{\overline{v}} v f_{2,n}(v)dv = E(v_{[2]})$ and $ER_2^{SSP} \to \int_0^{\overline{v}} v f_{2,n}(v)dv = E(v_{[2]})$ as $k \to 1$. $E(v_{[2]})$ is just the expectation of the second-highest of n independent random draws from $F(v)$. When $k = 1$, *individual demands are horizontal*, and the behavior in the second auction is unaffected by the outcome of the first auction. The high-valuation bidder will win both objects at a price of $v_{[2]}$, and revenue is the same in both auctions.

Given the increasing path of revenues over two straight second-price auctions, it is clear that the first of *two competitive sellers* has an incentive to change the auction format.[12] In this paper we shall restrict attention to the possible role of a buy-out price in the first auction when two competing sellers are selling identical objects. The first seller is interested in shifting revenues from the second to the first auction, while we shall also be interested in the consequences for efficiency and total revenue when the buy-out price is set optimally by the first seller. A mechanism may be inefficient in the present case, in the sense that it may (probabilistically) allocate an object to a bidder who would have received no object in an efficient mechanism. As we shall see this will be a feature of the mechanism for the case with two competing sellers where the first seller sets an optimal buy-out price.

3 Buy-Out Prices

Next, we allow the first seller to stipulate a buy-out price, B, and consider the augmented game described above. We first derive the relationship between the level of B and the valuations of bidders who will take this buy-out price. Then we look at the relationship between the buy-out price and the expected revenues to the two sellers, including how they are ranked. Finally, we demonstrate the revenue-enhancing effects of a suitably chosen buy-out price for the first seller and discuss the efficiency properties of the resulting sequence of auctions.

3.1 Equilibrium bidding strategies and expected revenues

We first look for a symmetric equilibrium in this augmented game in which bidders with valuations above some level $\widehat{v}$ take the buy-out price B in stage 1, while bidders with valuations below $\widehat{v}$ do not. In the augmented game, it is clear that if no bidder takes B, then it is common knowledge in equilibrium that all bidders have valuations below $\widehat{v}$. That is, beliefs are symmetric, and the logic of Proposition 1 (Black and de Meza (1992)) applies to the remainder of stage 1 and to stage 2. Hence, in stage 1 bids will be given by $b^1(v_i)$ in (1) with $v_i < \widehat{v}$, $\forall i = 1, 2, ..., n$. Further, regardless of how the good is sold in stage 1, it is well known that the bid in stage 2 will be kv_i if bidder i won the first auction, and v_i otherwise. In the following we suppress the subscript when this can be done without confusion.

In the equilibrium of the augmented game, a given value of B will induce a set $[\widehat{v}, \overline{v}]$ of bidder types to take the buy-out price B in stage 1. Changing B will change $\widehat{v}$. Hence, we can determine which $\widehat{v}$ to target, and chose B accordingly. Thus, we write $B(\widehat{v})$ as the value of B that induces bidder types above $\widehat{v}$ to take B in a symmetric equilibrium. Further, recall that $F_{1,m}(x) = F^m(x)$ is the distribution function of the first-order statistic in a sample of m, and $f_{1,m}(x) = \frac{dF^m(x)}{dx} = mF^{m-1}(x)f(x)$ is the density. Then, we can state the following result.

[11] Our general argument above for the n bidder case captures further aspects of Weber's results. With $k = 0$ (unit demands) bidding in both the first and the second auction is ultimately based purely on the expected *second*-highest value among a bidder's rivals, thus, on the *third-order* statistic $v_{[3]}$ of the n random valuations. From this it follows immediately that expected revenue is the same in the two auctions when $k = 0$ (cf. the observation in the text above).

[12] That is, short of moving to the last spot if possible. If selling-time is an endogenous variable, the two symmetric sellers might conceivably be involved in a war of attrition to become the last seller. This, however, is not the topic of this paper, and seller positions in the auction sequence are assumed to be exogenous.

Proposition 2 *Let $B(\widehat{v})$ be defined by*

$$\begin{aligned} B(\widehat{v})(1-F_{1,n}(\widehat{v})) &= \widehat{v}[1-F_{1,n}(\widehat{v})-n(1-F(\widehat{v}))F_{1,n-1}(\widehat{v}) \\ &\quad -\frac{1}{2}n(n-1)(1-F(\widehat{v}))(F(m(\widehat{v}))-F(\widehat{v}))F_{1,n-2}(\widehat{v})] \\ &\quad +n(n-1)(1-F((\widehat{v}))\int_0^{\widehat{v}}\left(kxF_{1,n-2}(kx)+\int_{kx}^{x}yf_{1,n-2}(y)dy\right)f(x)dx \\ &\quad +\frac{1}{2}n(n-1)(1-F(\widehat{v}))\int_{\widehat{v}}^{m(\widehat{v})}\left(kxF_{1,n-2}(kx)+\int_{kx}^{\widehat{v}}yf_{1,n-2}(y)dy\right)f(x)dx \end{aligned} \tag{4}$$

Then, a solution to (4) exists. Further, it is an equilibrium for bidders with $v \in [\widehat{v}, \overline{v}]$ to take the buy-out price $B(\widehat{v})$ in stage 1, and for bidders with $v \in [0, \widehat{v})$ not to.

Proof. See Appendix. ■

In the Web Appendix, we show that as $\widehat{v} \to \overline{v}$, $B(\widehat{v})$ approaches the expected value of the highest rival bid in the absence of a buy-out price. Generally, $B(\cdot)$ may not be monotonic, implying that for a given value of B, there could be multiple symmetric equilibria. Momentarily, we show that for any distribution and any $k \in (0,1)$, the first seller can strictly increase his revenue by offering a buy-out price that will be accepted with positive probability. In particular, it is shown that the first seller is better off if $\widehat{v}$ is sufficiently high (but strictly below $\overline{v}$). Since $B(\widehat{v})$ is continuous in $\widehat{v}$, the implication is that when B is sufficiently large, there is at least one symmetric equilibrium in which the first seller is better off. In fact, we prove in Section 4.5 that if B is set sufficiently high, then the first seller is better off in *any* symmetric equilibrium. Indeed, we argue that if there are multiple equilibria, it is plausible that the bidders will coordinate on the equilibrium which is preferred by the first seller. Finally, in Section 5 below, we solve explicitly for the optimal buy-out price in an example.

For completeness and for comparison with the revenues in Lemma 1, we now derive the expected revenue for arbitrary cut-offs.[13] Given some $\widehat{v}$, the equilibrium bidding strategies give rise to expected revenue stated as follows.

Proposition 3 *The expected revenue in the first auction is*

$$\begin{aligned} ER_1(\widehat{v}) &= \widehat{v}[1-F_{1,n}(\widehat{v})-n(1-F(\widehat{v}))F_{1,n-1}(\widehat{v})-\frac{1}{2}n(n-1)(1-F(\widehat{v}))F_{1,n-2}(\widehat{v})\left(F(m(\widehat{v}))-F(\widehat{v})\right)] \\ &\quad +n(n-1)\int_0^{\widehat{v}}\left(kvF_{1,n-2}(kv)+\int_{kv}^{v}yf_{1,n-2}(y)dy\right)(1-F(v)f(v)dv \\ &\quad +\frac{1}{2}n(n-1)(1-F(\widehat{v}))\int_{\widehat{v}}^{m(\widehat{v})}\left(kvF_{1,n-2}(kv)+\int_{kv}^{\widehat{v}}yf_{1,n-2}(y)dy\right)f(v)dv. \end{aligned} \tag{5}$$

Proof. Expected revenue in the first round is given as

$$ER_1(\widehat{v}) = B(\widehat{v})(1-F_{1,n}(\widehat{v})) + \int_0^{\widehat{v}} b^1(v)n(F(\widehat{v})-F(v))f_{1,n-1}(v)dv$$

since revenue is $B(\widehat{v})$ if the highest-valuation bidder has a valuation that exceeds $\widehat{v}$, and $b^1(v)$ if the second-highest type is v and the highest is somewhere between v and $\widehat{v}$. Substituting for $B(\widehat{v})(1-F_{1,n}(\widehat{v}))$ from (4), for $b^1(v)$ from (1), and noting that $f_{1,n-1}(v) = (n-1)F_{1,n-2}(v)f(v)$, expected revenue in the first round can be written as in (5). ■

In the Web Appendix, we characterize expected revenue in the second auction as well.

[13] This will be needed for the exposition of the example developed below in Section 5.

3.2 Optimal first-round buy-out prices and the path of revenues

We first turn to the main result of the paper, which states that it is always profitable for the first seller to introduce a suitably chosen buy-out price which is accepted with strictly positive probability, i.e., that he is better off targeting some $\widehat{v} < \overline{v}$. [14] Let v^* denote an optimal cut-off valuation and $B(v^*)$ the associated buy-out price.

Theorem 1 *An optimal buy-out price, $B(v^*)$, is accepted with positive probability, that is, $v^* < \overline{v}$.*

Proof. See Appendix. ■

Thus, the ability to offer a buy-out price strictly improves expected revenue to the first seller. In the proof of Theorem 1 it is established that the first seller is strictly better off with the introduction of a buy-out price if the cut-off, $\widehat{v}$, is sufficiently large. The *advantage* of a high buy-out price is that a relatively high revenue, $B(\widehat{v})$, is obtained if there is only one bidder with a high valuation (exceeding $\widehat{v}$). The *disadvantage* is that if two or more bidders happen to have very high valuations, the buy-out price effectively short-circuits the competition, and less revenue is obtained than if the bidders were to bid against each other in a standard auction. Since the seller is better off, the first effect must dominate the second.[15]

Recalling that prices are increasing in the bench-mark model, this leads to the question of whether the introduction of a suitably chosen buy-out price may even allow the first seller to overtake the second seller in terms of expected revenue. The following proposition answers this question in the negative, and we conclude that it is impossible for the first seller to "level the playing field" by introducing a buy-out price.

Proposition 4 (Increasing Prices) *Regardless of the buy-out price, expected revenue is strictly increasing over the sequence of auctions, that is, $ER_2(\widehat{v}) > ER_1(\widehat{v}), \forall \widehat{v} \in [0, \overline{v}]$.*

Proof. See Appendix. ■

3.3 Market efficiency

The use of a buy-out price by the first seller has implications for the overall market. In this section, we study how the allocation of objects is affected, and how overall revenue and revenue to the second seller change.

First, notice that when $n > 2$, *any* $\widehat{v} < \overline{v}$ has the potential to cause the final allocation to be inefficient. The reason is that the bidder with the third-highest valuation may win the first auction by accepting the buy-out price, which is clearly inefficient. Similarly, it is possible that the bidder with the second-highest valuation wins the first auction by accepting the buy-out price in cases where it is efficient for the bidder with the highest valuation to win both objects.

Of course, when $n = 2$ only the second possibility is relevant. In this case, observe that $\widehat{v} \geq k\overline{v}$ does not lead to inefficiency. If bidder 2, rather than bidder 1, wins stage one, we can conclude that $v_2 \geq \widehat{v} \geq k\overline{v} \geq kv_1$, so it is efficient for each bidder to win one unit. Hence, the buy-out price may lead to a change in the order in which units are won, but not to a change in the final allocation. However, if $\widehat{v} < k\overline{v}$, it may be efficient for the bidder with the highest type to win both objects, but the competing bidder may end up purchasing the first object by accepting the buy-out price.

[14] In the proof of this result, the reader should note that the assumptions needed for showing the usefulness of a buy-out price are more demanding for $n > 2$ than for $n = 2$. In particular, $f''(\cdot)$ is assumed to exist for the case with more than two bidders.

[15] As the existing literature has shown, this is never the case in a one-shot auction (absent risk-aversion and/or impatience). Hence, the *sequence* of auctions changes the relative magnitudes of the conflicting effects. For more on this, see Section 4.4 below.

For $n = 2$, Kirkegaard and Overgaard (2003) have shown that if $\widehat{v}$ is optimal, then it is strictly below $k\overline{v}$.[16] Therefore, the use of a buy-out price by the first seller leads to inefficiency.

Corollary 1 *The sequence of auctions is inefficient when the first seller sets the buy-out price optimally.*

Next, we argue that the introduction of a buy-out price by the first seller is detrimental to overall revenue. In order to do so, we impose the standard regularity assumption that the hazard rate, $\frac{f(v)}{1-F(v)}$, is increasing.

Proposition 5 *If demand is regular, any $\widehat{v} < \overline{v}$ leads to a strict loss in (expected) overall revenue, when $n > 2$. When $n = 2$, any $\widehat{v} < k\overline{v}$ (including an optimal $\widehat{v}$) leads to a strict loss in (expected) overall revenue, but overall revenue is unaffected if $\widehat{v} \geq k\overline{v}$.*[17,18]

Proof. See the Web Appendix. ■

Combining Theorem 1 and Proposition 5 implies that the second seller is worse off, as overall revenue diminishes, and the first seller is better off. However, recall that Proposition 4 implies that the second seller is better off than the first.

Corollary 2 *The second seller is worse off when the first seller sets an optimal buy-out price.*[19]

4 Discussion

In this section, we first provide an alternative, and very brief, proof that a buy-out price is profitable for the first seller when $n = 2$. Thereafter, we turn to a discussion of the modelling assumptions and the robustness of our results in the general case.

4.1 Buy-out prices with two bidders

Assuming, without loss of generality, that $v_1 > v_2$, the first auction is won by bidder 1 when there is no buy-out price, and with only two bidders it follows that the price in the second auction is $\min\{kv_1, v_2\}$. With a buy-out price, however, the first stage may be won by bidder 2, in which case revenue in the second stage is $kv_2 < \min\{kv_1, v_2\}$. Hence, as mentioned earlier, the second seller is worse off when the first seller offers a buy-out price.

Next, notice that if $\widehat{v} \geq k\overline{v}$, the final allocation is the same as without the buy-out price, although the order in which bidders win units may be reversed. Now, since the allocation is the same in the two different mechanisms, overall revenue must also be the same, by the *Revenue Equivalence Theorem*. As overall revenue is unchanged, and the second seller is worse off, the first must be better off. Consequently, it pays for the first seller to offer a buy-out price with $\widehat{v} \in [k\overline{v}, \overline{v})$. In fact, as noted above, if $\widehat{v}$ is optimal, then it is strictly below $k\overline{v}$.

[16] The example in Section 5 illustrates that this is not always the case when $n > 2$.

[17] For $n = 2$, Kirkegaard and Overgaard (2003) show that even if the regularity condition is violated, the sequence of auctions with a buy-out price yields lower revenue than without a buy-out price.

[18] Proposition 5 is a special case of a result in Kirkegaard (2004), where it is shown that any efficient mechanism is revenue superior to any mechanism which never awards a bidder two units more often than is efficient. This applies to the sequence of auctions with a buy-out price in stage one, since it may award a unit to a bidder who would not have received one in an efficient mechanism, and it never awards two units to a bidder who should have received none or one. In particular, it is evident from the preceding discussion that if the sequence is inefficient, it is because a bidder wins one unit, when he would have won none in an efficient mechanism.

[19] When $n = 2$, the second seller is worse off for *any* $\widehat{v} < \overline{v}$. See Kirkegaard and Overgaard (2003) for a formal proof, or see Section 4 below for an intuitive explanation. When $n > 2$, however, it is easy to construct examples in which the second seller is better off when the first seller uses a *non-optimal* buy-out price.

4.2 English auctions

Formally, we have modelled the two auctions as second-price, *sealed-bid* auctions. This allows us to focus on the role of buy-out prices and economize somewhat on the discussion of the auctions, since a strategy in the first auction is simply a bid.

In contrast, a strategy in the first of two *English* auctions is more complex, as it specifies when to stop bidding, contingent on the information at the time, which may include whether and when other bidders have dropped out. Février, Roos and Visser (2005) derive the equilibrium strategies in this case.

Without a buy-out price, the sequence of auctions are revenue equivalent stage by stage, regardless of whether they are modelled as second-price, sealed-bid auctions or English auctions.

Similarly, with a buy-out price, it can be shown that the payoff to a type $\widehat{v}$ bidder from accepting the buy-out price is the same for either auction format, and so is the payoff from rejecting. Hence, $B(\widehat{v})$ does not depend on the auction format. Likewise, for given $\widehat{v}$, the expected revenue in the two auctions are independent of the auction format.

Alternatively, since we motivated our analysis with explicit reference to *eBay*, we could simply argue that a second-price, sealed-bid format is the descriptively most relevant formalization. To see this, note that *eBay* has a *firm* closing time. That the closing time is firm means that the auction ends no later than at some pre-announced time T. This, in turn, implies that the end-game is one in which bidders will ultimately have to choose their final (proxy) bids simultaneously as in a sealed-bid format despite the appearance of the auction as an open, ascending-bid format. The second-highest of these simultaneous bids will determine the trading price. This is in sharp contrast to other online auction sites (e.g., *Yahoo!*), where the closing time is *soft*, which means that the pre-announced closing time T is automatically extended for a certain period as long as competitive bids keep arriving. Thus, when the closing time is soft, any bidder always has the opportunity to respond to new bids as in an open, ascending-bid format.

4.3 Permanent buy-out prices

In this paper we have chosen to focus on *eBay*'s version of the buy-out price. The buy-out price is temporary on *eBay*, whereas *Yahoo!* offers a permanent buy-out price (termed the *Buy Price*). Reynolds and Wooders (2006) compare the two types of buy-out prices[20] one-shot auctions, and their findings are outlined in the following two paragraphs.

With a permanent buy-out price the auction can be thought of as an English auction, where, at any time, a bidder can choose to accept the buy-out price, rather than continuing the bidding process. Then, bidders with very high valuations may accept the buy-out price immediately. Bidders with lower valuations initially ignore the buy-out price, but as bidding in the English auction progresses, they become more and more pessimistic about the severity of the competition and eventually accept the buy-out price (given that it is lower than the valuation). The higher the valuation, the sooner the buy-out price is accepted.

If the buy-out price is very high, not even a bidder with valuation $\overline{v}$ will accept it immediately. As the price in the English auction increases, however, the buy-out price may be accepted. If so, it is accepted by the bidder with the highest valuation, and the auction is efficient. On the other hand, if the buy-out price is such that it would be accepted by a bidder with valuation in the interval $[\widehat{v}, \overline{v}]$ in an *eBay* auction, the same bidder accepts it immediately in the *Yahoo!* auction. If it is not accepted immediately, it may be accepted later on, by the high-valuation bidder. Hence, $B(\widehat{v})$ causes the same type of inefficiency whether it is a permanent or temporary buy-out price.

In fact, by the *Revenue Equivalence Theorem*, the *eBay* and *Yahoo!* auction formats yield the same revenue, for a given $B(\widehat{v})$. Thus, *our* results are also valid when the buy-out price is permanent.

[20] See also Hidvégi, Wang and Whinston (2006) for an analysis of permanent buy-out prices.

More generally, we conjecture that the results of this paper are robust to small changes in the extensive form of the game. The reason is straightforward, and relies only on the possibility that the buy-out price may affect who wins the first auction. Specifically, consider the case $n = 2$ and assume that in the equilibrium of the particular game, there is a $\widehat{v}$ such that stage one is won by the bidder with the highest valuation, if at most one bidder has a valution that exceeds $\widehat{v}$, but that there is a strictly positive probability that it is won by the other bidder otherwise. In this event, the second seller is worse off (because competition is diminished in stage two). At the same time, however, we know that overall revenue is unchanged if $\widehat{v} \geq k\overline{v}$ (because the sequence of auctions is efficient in this case). Consequently, when $\widehat{v}$ is sufficiently high, the first seller is better off with a buy-out price as long as the identity of the winner in stage one changes with positive probability.

4.4 Unit demands or horizontal demands

So far, we have focused on the case where $k \in (0,1)$, for the simple reason that a buy-out price cannot be explained in the model presented here when $k \in \{0,1\}$. In this section, we explain why it is necessary that bidders have multi-unit demands ($k > 0$) and that marginal utility is decreasing ($k < 1$). To do so, we combine Proposition 5, which also holds for $k \in \{0,1\}$, with the observation, to follow, that the second seller is better off when a buy-out price is employed, for $k \in \{0,1\}$. Consequently, the first seller must be worse off should he choose to offer a buy-out price if either $k = 0$ or $k = 1$.

First, consider the case where bidders desire only one unit, or $k = 0$. Arranging bidders in descending order, $v_1 > v_2 > ... > v_n$, bidder 1 wins the first stage for sure in the sequence of auctions without a buy-out price, and the price in stage two is v_3, since bidder 2 outbids bidder 3. With a buy-out price, the price in the second auction remains v_3, if the first auction is won by either bidder 1 or bidder 2. Otherwise, however, the price in the second auction will be v_2, as bidder 1 will outbid bidder 2 at this stage. Hence, the second seller is better off, while overall revenue decreases, by Proposition 5.

Second, if marginal utility is constant, or $k = 1$, revenue in the second auction is v_2 independently of the outcome in the first auction, and by Proposition 5 it follows that the first seller is worse off, should he decide to offer a buy-out price. Alternatively, this can be interpreted as saying that for the first seller to offer a buy-out price, *it is necessary that there are future auctions*. The reason is that when $k = 1$, bidder behavior in the second auction is unaffected by the outcome of the first auction, so when deciding on a strategy in the first auction, bidders need not take into consideration that there are future auctions. Hence, the first auction might as well be a one-shot auction. In one-shot auctions with risk neutral and patient bidders, it is well known that the second price sealed bid auction maximizes expected revenue absent reserve prices, and there is no reason to modify it with a buy-out price.

4.5 Multiple equilibria

As mentioned above, it is possible that $B(\widehat{v})$ is associated with multiple equilibria, with different cut-offs. To illustrate, let x and y, with $0 < x < y \leq \overline{v}$, denote two cut-offs associated with different symmetric equilibria where $B(x) = B(y)$. Then, the first seller is *better off* if the cut-off is x rather than y. The reason is that since $x < y$, the buy-out price is accepted *more often* in the equilibrium with cut-off x. This is advantageous to the first seller, since the buy-out price is relatively high compared to what a bidder with valuation between x and y would otherwise expect to pay contingent on winning the first object.[21] Hence, it is beneficial to the first seller that the buy-out price is exercised more often.

Proposition 6 (Multiple Equilibria) *If $B(x) = B(y)$ and $0 < x < y \leq \overline{v}$ then $ER_1(x) > ER_1(y)$. That is, if there are multiple equilibria associated with any given B, the first seller prefers the equilibrium in which the buy-out price is accepted more often.*

[21] This is made precise in the proof of Proposition 6 below.

Proof. See the Web Appendix. ■

Recall that when we proved that the first seller benefits from introducing a buy-out price (see proof of Theorem 1), we did so by showing that lowering $\widehat{v}$ marginally from $\overline{v}$ increases expected revenue in the first auction. In other words, we showed that *high* values of $\widehat{v}$ are profitable. The above proposition adds to Theorem 1 by showing that if there are multiple equilibria, then lower values of $\widehat{v}$ are *even more* profitable. In conclusion, the *least* profitable equilibrium (the highest cut-off) is better for the first seller than the equilibrium without a buy-out price. Thus, there are values of the buy-out price, B, for which *all* equilibria are *more* profitable for the first seller than no buy-out price.

Thus, the first seller prefers the equilibrium with the *lowest* cut-off. Moreover, a subset of the bidders feel the same way, and we now argue that bidders are likely to play this particular equilibrium. So, suppose two equilibria with cut-offs x and y, with $x < y$, are compared. We first notice that bidders with valuations below x are indifferent between the two, because their chance of winning and their expected payments are unaffected. However, a bidder with a valuation slightly above x would either be indifferent or strictly prefer the first equilibrium. The reason is that he wins at least as often in the first equilibrium, since it is possible that by accepting the buy-out price he will win an object when otherwise he would have won none. It is a general implication of the Revenue Equivalence Theorem that bidders prefer mechanisms in which they win more often. Thus, when bidders coordinate on an equilibrium, any buyer with valuation close to x would "lobby" for the first equilibrium. In this sense, the interests of the first seller and the bidders are aligned. We could even argue that the "y-equilibrium" would be unravelled by "low"-valuation bidders.[22,23]

5 Example

To add some further insights into the results above, in this section we consider the case where the first-unit valuations are *uniformly* distributed on the unit interval, that is, $\overline{v} = 1$, $f(v) = 1$ and $F(v) = v$. We concentrate on the case with two bidders, while remarking briefly on the additional insights which can be obtained when more than two bidders are involved. The full details of the example can be found in the Web Appendix.

Two bidders

Assume first that there is *no buy-out* price. Bidding in the first round as captured by (1) reduces to $b^1(v) = kv$, where we note that $b^1(v) \to 0$ for $k \to 0$ (*single-unit* demand),[24] $b^1(v) \to v$ for $k \to 1$ (*horizontal* demand).

Now, consider introducing a *buy-out* price in the example. In Fig. 1 the optimal cut-off valuation, v^*, and the associated buy-out price, $B(v^*)$, are plotted against k.

[22] Alternatively, and in line with much of the literature on mechanisms, contracts and principal-agent relations, we could simply assume directly that bidders play the equilibrium preferred by the first seller ("the mechanism designer").

[23] Finally, forward-induction logic might eliminate unprofitable equilibria from the perspective of the first seller. That is, each bidder reasons that the first seller would only have employed the buy-out option, if he anticipated bidders to play a continuation equilibrium which makes him no worse off.

[24] This is a special case of an example developed by Krishna (2002, Example 15.2, p. 219).

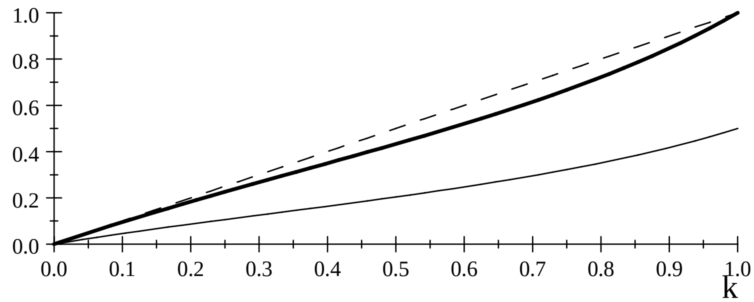

Fig. 1: Optimal cut-off and buy-out
(v^* - *fat*, $B(v^*)$ - *thin*, 45^0 - *dashed*)

Next, Fig. 2 illustrates how expected revenues in the two auctions depend on k, when the buy-out price is chosen optimally by the first seller, that is $ER_1(v^*)$ and $ER_2(v^*)$. For comparison the figure also plots expected revenues without a buy-out price, ER_1^{SSP} and ER_2^{SSP}.

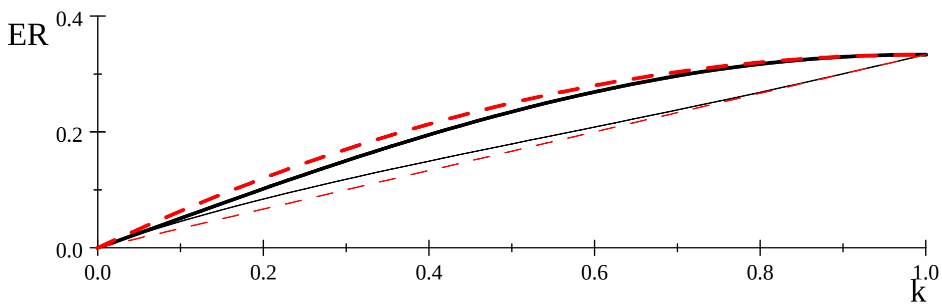

Fig. 2. Comparison of auction revenues
(ER_1^{SSP} - *thin dashed*, ER_2^{SSP} - *fat dashed*, $ER_1(v^*)$ - *thin*, $ER_2(v^*)$ - *fat*)

Finally, let $G = 100 \times \frac{ER_1(v^*) - ER_1^{SSP}}{ER_1^{SSP}}$ denote the percentage gain to the first seller from an optimally chosen buy-out price compared to the straight second-price auction. Then the Table 1 captures central the features of the example.

k	ER_1^{SSP}	v^*	$B(v^*)$	$ER_1(v^*)$	G
0.01	0.00333	0.00995	0.00495	0.00495	48.65
0.10	0.03333	0.09549	0.04597	0.04558	36.75
0.25	0.08333	0.22618	0.10623	0.10176	22.12
0.50	0.16667	0.43308	0.20404	0.17931	7.58
0.75	0.25000	0.66667	0.32222	0.25309	1.24

Table 1

The last column and Fig. 2 reveal that the value from the perspective of the first seller of introducing a buy-out price is substantial when the individual demand functions are relatively steep (k small). When demands are steep, and there are only two bidders, the competition for the first object will be weak. It follows that the first seller has a strong incentive to try to improve his position in this case by introducing a suitably chosen buy-out price.

Recall that revenue equivalence and efficiency is lost when $\widehat{v}$ is set below $k = k\overline{v}$. Hence, a comparison of the first and third column is indicative of the inefficiency when $\widehat{v}$ is set optimally. For example, when

$k = k\overline{v} = \frac{1}{2}$ the optimal $\widehat{v}$ is aproximately 0.43, which implies that there is a small, but "non-trivial", probability that the final allocation is inefficient. Note that $k = \frac{1}{2}$ implies that $ER_2^{SSP} - ER_1^{SSP} = \frac{1}{3}k(1-k)$ is maximized. When the first seller sets the optimal buy-out price $B(v^*) \approx 0.2$, he manages to increase his expected revenue by 7.58%, while total revenue falls by only 0.58%.

As mentioned earlier, $B(v)$ need not necessarily be monotonic in v. Indeed, when $n = 2$, the buy-out price is globally increasing in v when $k > \sqrt{2} - 1$, but *not* when k is smaller (see the Web Appendix for details). Hence, when k is large, the optimal buy-out price is associated with a unique cut-off valuation, that is, the optimal cut-off valuation.

On the other hand, Fig. 3 illustrates the relationship between the cut-off and the buy-out price when $k = \frac{1}{4}$. The optimal cut-off is approximately 0.226, which necessitates a buy-out price of approximately 0.106. However, at such a buy-out price there are, in fact, *three* symmetric equilibria (cut-off points). Specifically, in addition to the optimal cut-off, there are two larger cut-offs that also satisfy the equation for $B(v)$ in (4). Nevertheless, since expected revenue to the first seller is *single-peaked* in the example, *any* of these equilibria or cut-offs yield strictly higher revenue than no buy-out price. In fact, any equilibrium associated with any buy-out price in excess of 0.106 will increase expected revenue to the first seller.

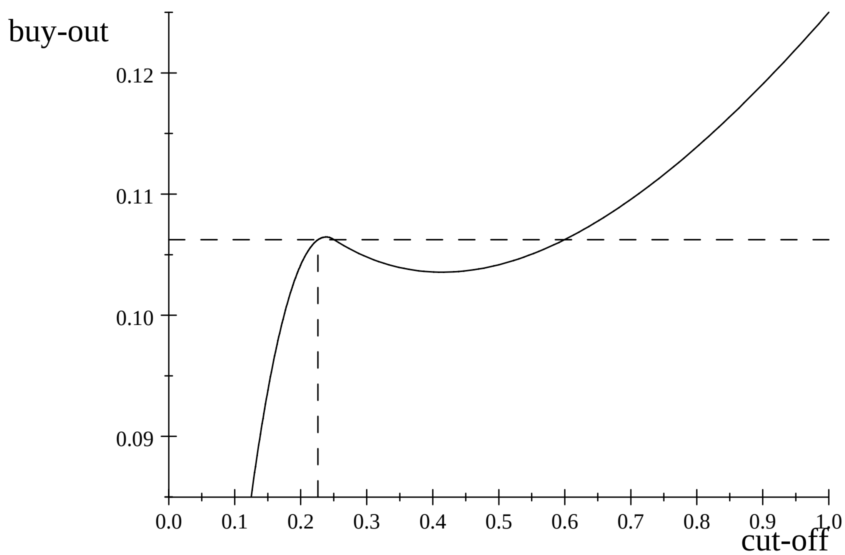

Fig 3. Buy-out prices and cut-off valuations, $k = \frac{1}{4}$.

More than two bidders

To capture the added features when there are multiple bidders, consider the case with $n = 3$. In this case, the optimal cut-off is "U-shaped". For small values of k, the optimal value of $\widehat{v}$ is above k, and is decreasing until k reaches $\overline{k} \simeq 0.60149$. Indeed, for $k = 0$, the optimal value of $\widehat{v}$ is one, implying that the buy-out is never exercised. The reason is that when $k = 0$, the sequence of (straight) auctions is optimal, and any changes to the design would make the first seller worse off. After $k > \overline{k}$, the optimal value of $\widehat{v}$ starts to increase. For $k = 1$, the optimal cut-off is 1, implying again that the buy-out price is never accepted. Again, the reason is that with $k = 1$, the sequence of (straight) auctions is optimal. The closer k is to zero or one, the closer the sequence of auctions is to being optimal (overall and for seller 1), and the less incentive there is to manipulate the auction format. That explains why v^* is close to one when k is close to either zero or one. Consequently, neither the optimal cut-off nor the probability that the buy-out price is accepted are monotonic in k.[25] Fig. 4 illustrates the optimal optimal cut-off,

[25] The "U-shape" of the optimal cut-off for $n > 2$ generalizes to any distribution of valuations, $F(v)$, for which $ER_1(\widehat{v})$ is

v^*. For comparison, the figure also repeats the optimal cut-off when $n = 2$. When $n = 2$, we recall that the cut-off is everywhere increasing in k and below $\widehat{v} = k$. We note that the addition of another bidder causes the optimal cut-off to increase. However, since there are more bidders, this does not necessarily mean that it becomes less likely that the buy-out is accepted.

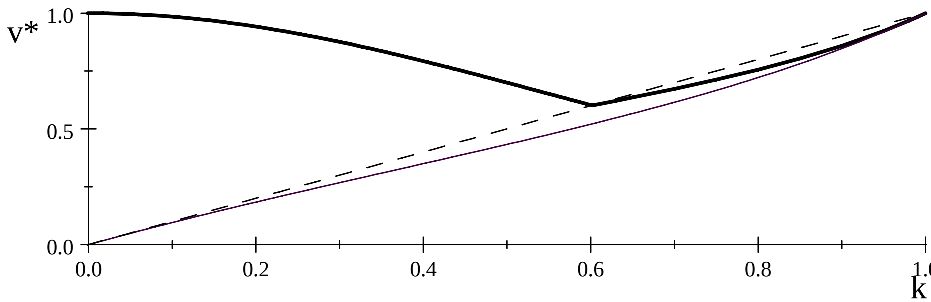

Fig. 4: Optimal cut-offs
($n = 3$ - *fat*, $n = 2$ - *thin*, 45^0 - *dashed*)

As in the preceeding case with two bidders, we can also capture central features of the example in an alternative way when there are three bidders.

k	ER_1^{SSP}	v^*	$ER_1(v^*)$	G
0.20	0.26000	0.94231	0.26000	≈ 0
0.40	0.29000	0.79310	0.29018	0.06
0.60	0.34000	0.60294	0.34282	0.83
0.80	0.41000	0.75587	0.41252	0.61
0.90	0.45250	0.85878	0.45311	0.13

Table 2

From the last column of Table 2, we note that the gain to the first seller from an optimally chosen buy-out price is small for any k when $n = 3$ compared to the case where $n = 2$. Also note that the gain is non-monotonic in k.

Discussion

In the Web Appendix, we examine in detail the efficiency loss associated with a buy-out price in the first auction. With two bidders the loss is miniscule, about 0.01% for most values of k. Interestingly, with an additional bidder the efficiency loss *increases*. Though the loss is still small, it is an order of magnitude larger than with only two bidders. Notice that when $n > 2$, *any* $\widehat{v}$ below $\overline{v}$ gives rise to potential inefficiency. As illustrated by Fig. 4, the optimal cut-off is far below $\overline{v}$ for $n = 3$ and intermediate values of k. On the other hand, for $n = 2$, inefficiency arises only if $\widehat{v}$ is below $k\overline{v}$. However, from Fig. 4 it is evident that v^* is only slightly below $k\overline{v}$ for $n = 2$. Finally, as the number of bidders increases without bound (intensive bidder competition), the efficiency loss clearly disappears. Thus, we conclude that the efficiency loss, when the buy-out price in the first auction is set optimally, is *non-monotonic* in the number of bidders.

Based on this example, it may be argued that a buy-out price does not raise revenue much when there are 3 or more bidders, and that in most *eBay* auctions, say, there are many bidders. However, it is also typically the case that there more than 2 items for sale on *eBay*. The main import of the example is to suggest that the *ratio* of goods to bidders is key to determining how profitable a buy-out price might be. It could also be argued from the example that even though the percentage gain is sometimes

single-peaked in $\widehat{v}$ for all k.

large when there are 2 bidders, absolute gains are small. This fact is, however, due to the (conventional) normalization of the support to [0,1]. For example, when $k = 0.25$, revenue in the first auction is 0.08333 without the buy-out price and 0.10176 with the buy-out price. If the upper end-point of the support had been 1000 rather than 1, these numbers would have changed to 83.33 and 101.76, respectively. As long as $k > 0$, the numbers can be made arbitrarily large by expanding the interval. Hence, a fixed fee (of x dollars, say) for using the buy-out facility does not necessarily wipe out the potential, absolute gain to the seller. We fully acknowledge, though, that buy-outs are less useful to the first seller when demand is strong compared to supply (k large or n large). But, most attempts to manipulate revenue (through buy-outs, participation fees, reserve prices, etc.) are bound to be less effective when bidder competition is intensive.

6 Concluding Remarks

In this paper we sought to explain the use of buy-out prices by observing that online auction markets are dynamic, with players knowing that goods not presently on the market are likely to be offered in the future and with buyers displaying downward-sloping, multi-unit demands. It was shown that there is an incentive for an early seller to offer a buy-out price that is accepted with positive probability. Suitably chosen, such a buy-out price will increase revenue in the early auction. As shown by example, the revenue gain to the first seller from using the buy-out facility is largest when demand is weak, in the sense of either few bidders or steep demands. In any case, revenue in subsequent auctions will decrease, as will the sum of revenues. When the buy-out price in the early auction is chosen optimally, the resulting string of auctions is inefficient, in the sense that the item on offer may go to a buyer who would have received none in any efficient selling mechanism. By example, we showed that the potential efficiency loss can be non-monotonic in the number of bidders (that is, the intensity of bidder competition).

A couple of issues could be mentioned before we close. *First*, we assume throughout that the auction sequence is fixed, while we show that sellers would prefer to go last (with or without the buy-out). Thus, there is a certain tension, but similar concerns relate to any kind of auction sequence that does not have the martingale property. Any such sequence would be vulnerable to the critique that some "competing" seller would try to change place or massage the auctions until the martingale property is restored. If the sequence had a declining price path, sellers would rush to be first, and if it had an increasing path, sellers would attempt to outwait each other. So, we should observe only *revenue equivalent* simultaneous or sequential auctions. Of course, the sequence could be determined by unmodelled elements of seller payoffs. This might include a need to raise cash for exogenous reasons or shocks to preferences, which precludes Seller 1 from waiting, or sellers might be middlemen with inventory or commissions to sell arriving randomly. The latter might be most relevant for perishables and goods with holding costs. We could also appeal to different degrees of impatience (or discounting) among sellers. This might in itself fully determine the sequence (the game form), but the first seller would still attempt to adapt the basic game form to his own advantage by introducing a buy-out. *Secondly*, by reference to our leading example, since overall revenue decreases (and with it probably *eBay*'s earnings), it appears puzzling that *eBay* offers the buy-out price in the first place. In effect, we have ignored the role of this third type of player (the auction house). However, *eBay* does not hold a monopoly on online auctions. Competitive pressure might be important in understanding why *eBay* introduced the buy-out price (sellers can find other sites that have the buy-out facility). As documented by Lucking-Reiley (2000), the market leader *eBay* was not the first to offer the buy-out option. Indeed, Reynolds and Wooders (2006) describe how *Yahoo!* introduced the feature first, and *eBay* followed not long thereafter. These and other industrial organization features of online auction markets form interesting topics for further research.

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Appendix

Proof of Proposition 2. We note that if the buy-out price is not accepted, the bid in the first auction will be equal to the expected value of the bid in the second auction, given the first auction is lost to a bidder with the same valuation. Thus, bidding proceeds according to

$$b^1(x) = kx\frac{F_{1,n-2}(kx)}{F_{1,n-2}(x)} + \int_{kx}^{x} y\frac{f_{1,n-2}(y)}{F_{1,n-2}(x)}dy$$

which implies that

$$b^1(x)F_{1,n-2}(x) = kxF_{1,n-2}(kx) + \int_{kx}^{x} yf_{1,n-2}(y)dy$$

This enables us to derive the payoff from taking or not taking $B(\widehat{v})$, given the bidder has valuation $\widehat{v}$.

First, by accepting the buy-out price, expected payoff is

$$\begin{aligned} EU(B,\widehat{v}) &= (\widehat{v} - B(\widehat{v}))\Pr(W|\widehat{v}) + \int_0^{k\widehat{v}} (k\widehat{v} - x)f_{1,n-1}(x)dx \\ &\quad + \frac{1}{2}(n-1)\int_{\widehat{v}}^{m(\widehat{v})} \left((\widehat{v} - kx)F_{1,n-2}(kx) + \int_{kx}^{\widehat{v}} (\widehat{v} - y)f_{1,n-2}(y)dy \right) f(x)dx. \end{aligned}$$

The *first* term is the expected payoff from the first auction. In equilibrium, the bidder takes the buy-out $B(\widehat{v})$, as does every rival with valuation above $\widehat{v}$. If more than one bidder take $B(\widehat{v})$, the winner is determined by a fair lottery. Hence, the probability of winning when taking the buy-out price, $\Pr(W|\widehat{v})$, can be written as

$$\begin{aligned} \Pr(W|\widehat{v}) &= \sum_{i=0}^{n-1} \binom{n-1}{i} (1 - F(\widehat{v}))^i F(\widehat{v})^{n-1-i} \frac{1}{i+1} \\ &= \frac{1}{n(1 - F(\widehat{v}))} \sum_{i=0}^{n-1} \frac{n \times (n-1)!}{(i+1)!(n-(i+1))!} (1 - F(\widehat{v}))^{i+1} F(\widehat{v})^{n-(i+1)} \\ &= \frac{1}{n(1 - F(\widehat{v}))} \sum_{j=1}^{n} \binom{n}{j} (1 - F(\widehat{v}))^j F(\widehat{v})^{n-j} = \frac{1 - F_{1,n}(\widehat{v})}{n(1 - F(\widehat{v}))} \end{aligned}$$

where the last equality follows by applying the Binomial Theorem. The *second* term captures the possibility that the bidder wins both auctions. Finally, the *third* term comes from the fact that the bidder may lose the first auction, but win the second. If he loses the first auction, the winner must have a valuation above $\widehat{v}$. If there are several rivals with valuation above $\widehat{v}$, the bidder will also lose the second auction (since he, by assumption, has valuation $\widehat{v}$). Thus, only if there is precisely one bidder among the $n-1$ rivals, is it possible to lose stage one (which then happens with probability 0.5) and subsequently win stage two. This, however, requires that the rival has type below $m(\widehat{v})$, such that his bid in stage two will be below $\widehat{v}$. The price in stage two is clearly the maximum of k times the winner's valuation and the highest of the other rivals' valuations.

If the bidder with valuation $\widehat{v}$ decides not to take $B(\widehat{v})$, it is optimal for him to outbid everybody else in the first auction, if the buy-out price is not accepted by any of the rivals. By doing so, he ends up paying the bid of his strongest rival, $b^1(v)$ say, with $v \leq \widehat{v}$, and possibly also winning stage two. If he does not outbid his rivals, he will win stage two at an expected price of

$$b^1(v) = kv\frac{F_{1,n-2}(kv)}{F_{1,n-2}(v)} + \int_{kv}^{v} y\frac{f_{1,n-2}(y)}{F_{1,n-2}(v)}dy$$

Hence, if the bidder with valuation $\widehat{v}$ does not take $B(\widehat{v})$, his expected payoff is

$$\begin{aligned} EU(NB,\widehat{v}) &= \int_0^{\widehat{v}} (\widehat{v} - b^1(x)) f_{1,n-1}(x)dx + \int_0^{k\widehat{v}} (k\widehat{v} - x) f_{1,n-1}(x)dx \\ &\quad +(n-1)\int_{\widehat{v}}^{m(\widehat{v})} \left((\widehat{v} - kx)F_{1,n-2}(kx) + \int_{kx}^{\widehat{v}} (\widehat{v} - y) f_{1,n-2}(y)dy \right) f(x)dx \end{aligned}$$

The first term comes from the fact that the bidder outbids everybody else in the first auction, if no one accepted $B(\widehat{v})$. The second and third terms are similar to those in $EU(B,\widehat{v})$, with the exception that the first stage is now lost with probability one, if there is a rival with valuation above $\widehat{v}$.

As we are looking for an equilibrium in which $B(\widehat{v})$ is accepted if, and only if, the bidder has valuation above $\widehat{v}$, it must be the case that the bidder with valuation $\widehat{v}$ is indifferent between accepting and not accepting. This indifference condition gives rise to (4). The details are in the Web Appendix. The Web Appendix also contains the proof of the fact that bidders with valuation different from $\widehat{v}$ will not deviate from the equilibrium strategy. ■

Proof of Theorem 1. To show this result, we focus on high values of $\widehat{v}$, by assuming that $\widehat{v} > k\overline{v}$, or $m(\widehat{v}) = \overline{v}$. First, for given $\widehat{v} > k\overline{v}$ we can then write the first-round revenue as

$$\begin{aligned} ER_1(\widehat{v}) &= \widehat{v}[1 - F^n(\widehat{v}) - nF^{n-1}(\widehat{v})(1 - F(\widehat{v})) - \frac{1}{2}n(n-1)(1 - F(\widehat{v}))^2 F^{n-2}(\widehat{v})] \\ &\quad +n(n-1)\int_0^{\widehat{v}} \left(kxF^{n-2}(kx) + \int_{kx}^{x} y(n-2)F^{n-3}(y)f(y)dy \right) (1 - F(x)f(x)dx \\ &\quad +\frac{1}{2}n(n-1)(1 - F(\widehat{v}))\int_{\widehat{v}}^{\overline{v}} \left(kxF^{n-2}(kx) + \int_{kx}^{\widehat{v}} y(n-2)F^{n-3}(y)f(y)dy \right) f(x)dx \end{aligned}$$

After some manipulation, the derivative with respect to $\widehat{v}$ reduces to

$$\begin{aligned} ER_1'(\widehat{v}) &= 1 - F^n(\widehat{v}) - nF^{n-1}(\widehat{v})(1 - F(\widehat{v})) - \frac{1}{2}n(n-1)(1 - F(\widehat{v}))^2 F^{n-2}(\widehat{v}) \\ &\quad -\frac{1}{2}n(n-1)f(\widehat{v})\int_{\widehat{v}}^{\overline{v}} \left(kxF^{n-2}(kx) + \int_{kx}^{\widehat{v}} y(n-2)F^{n-3}(y)f(y)dy \right) f(x)dx \\ &\quad +\frac{1}{2}n(n-1)(1 - F(\widehat{v}))f(\widehat{v}) \left(k\widehat{v}F^{n-2}(k\widehat{v}) + \int_{k\widehat{v}}^{\widehat{v}} y(n-2)F^{n-3}(y)f(y)dy \right) \end{aligned}$$

In the special case where $n = 2$, the first line in equal to zero, and the rest is strictly negative when $\widehat{v} \in [k\overline{v}, \overline{v})$, implying that $\widehat{v} = k\overline{v}$ is better than any higher $\widehat{v}$. When $n \geq 3$, the first line is non-negative, since it equals

$$1 - \sum_{i=0}^{2} \binom{n}{i} (1 - F(\widehat{v}))^i F^{n-2}(\widehat{v})$$

while the rest is non-positive. Hence, it is difficult to generally determine the sign of $ER_1'(\widehat{v})$. However, it is easily seen that $ER_1'(\overline{v}) = 0$.

The second derivative with respect to $\widehat{v}$ equals

$$\begin{aligned} ER_1''(\widehat{v}) &= -\frac{1}{2}n(n-1)(n-2)(1-F(\widehat{v}))^2F^{n-3}(\widehat{v})f(\widehat{v}) + \frac{1}{2}n(n-1)(1-F(\widehat{v}))f(\widehat{v})kF^{n-2}(k\widehat{v}) \\ &\quad -\frac{1}{2}n(n-1)f'(\widehat{v})\int_{\widehat{v}}^{\overline{v}}\left(kxF^{n-2}(kx)+\int_{kx}^{\widehat{v}}y(n-2)F^{n-3}(y)f(y)dy\right)f(x)dx \\ &\quad +\frac{1}{2}n(n-1)(1-F(\widehat{v}))f'(\widehat{v})\left(k\widehat{v}F^{n-2}(k\widehat{v})+\int_{k\widehat{v}}^{\widehat{v}}y(n-2)F^{n-3}(y)f(y)dy\right) \end{aligned}$$

Again, $ER_1''(\overline{v}) = 0$. However, it is easy to evaluate the third derivative at $\overline{v}$,

$$ER_1'''(\overline{v}) = -\frac{1}{2}n(n-1)(f(\overline{v}))^2kF^{n-2}(k\overline{v}) < 0.$$

This implies that immediately to the left of $\overline{v}$, $ER_1''(\cdot)$ is positive, which, in turn, means that $ER_1'(\cdot)$ is negative just to the left of $\overline{v}$. It follows that the first seller is better off with some cut-off strictly less than $\overline{v}$ than without a cut-off. This completes the proof. ■

Proof of Proposition 4. We can write expected revenue in the first auction as

$$ER_1(\widehat{v}) = \int_0^{\widehat{v}} b^1(x)n(n-1)(F(\widehat{v})-F(x))F^{n-2}(x)f(x)dx + B(\widehat{v})(1-F^n(\widehat{v}))$$

or, after inserting $B(\widehat{v})$, as

$$\begin{aligned} ER_1(\widehat{v}) &= \int_0^{\widehat{v}} b^1(x)n(n-1)(F(\widehat{v})-F(x))F^{n-2}(x)f(x)dx \\ &\quad +\int_0^{\widehat{v}} b^1(x)n(n-1)(1-F(\widehat{v}))F^{n-2}(x)f(x)dx \\ &\quad +\int_{\widehat{v}}^{m(\widehat{v})}\left(kxF_{1,n-2}(kx)+\int_{kx}^{\widehat{v}}yf_{1,n-2}(y)dy\right)\frac{1}{2}n(n-1)(1-F(\widehat{v}))f(x)dx \\ &\quad +\widehat{v}\left[1-\left(F^n(\widehat{v})+n(1-F(\widehat{v}))F^{n-1}(\widehat{v})+\frac{1}{2}n(n-1)(1-F(\widehat{v}))F^{n-2}(\widehat{v})\left(F(m(\widehat{v}))-F(\widehat{v})\right)\right)\right] \end{aligned}$$

Now, in order to compare $ER_1(\widehat{v})$ with expected revenue in the second auction, we identify four possible events.

i) All n buyers have valuations below $\widehat{v}$. The probability of this event is $F^n(\widehat{v})$, or

$$\int_0^{\widehat{v}} n(n-1)(F(\widehat{v})-F(x))F^{n-2}(x)f(x)dx$$

Notice the relationship with the first term in $ER_1(\widehat{v})$.

ii) Precisely one buyer has a valuation above $\widehat{v}$. The probability of this event is $n(1-F(\widehat{v}))F^{n-1}(\widehat{v})$. Again, notice the relationship with the second term in $ER_1(\widehat{v})$.

iii) Precisely two buyers have valuations above $\widehat{v}$, but whoever (of the two) wins stage one has a valuation not exceeding $m(\widehat{v})$, implying that his bid, and thus the price, in stage two will be below $\widehat{v}$. If $m(\widehat{v}) = \overline{v}$, the probability of this event is

$$n(n-1)\frac{1}{2}F^{n-2}(\widehat{v})(1-F(\widehat{v}))^2$$

which is the probability mass in the third term in $ER_1(\widehat{v})$, when $m(\widehat{v}) = \overline{v}$. When $m(\widehat{v}) = \widehat{v}/k$, the probability is

$$n(n-1)F^{n-2}(\widehat{v})(1-F(m(\widehat{v})))(F(m(\widehat{v}))-F(\widehat{v}))\frac{1}{2} - n(n-1)\frac{1}{2}F^{n-2}(\widehat{v})(F(m(\widehat{v}))-F(\widehat{v}))^2$$

This is the probability that the buyer with the highest valuation has a valuation above $m(\widehat{v})$, yet loses to a buyer with a valuation in the range $\widehat{v}$ to $m(\widehat{v})$ (the probability of which is 0.5), less the probability that both of the buyers who are willing to accept $B(\widehat{v})$ have valuations in the range from $\widehat{v}$ to $m(\widehat{v})$. Notice that this probability can be rewritten as

$$\frac{1}{2}n(n-1)F^{n-2}(\widehat{v})(1-F(\widehat{v}))(F(m(\widehat{v}))-F(\widehat{v}))$$

which is easily seen to be the probability mass in the third term of $ER_1(\widehat{v})$.

iv) All other possibilities not included in i), ii) and iii). The probability of this is clearly the term in brackets in the fourth term in $ER_1(\widehat{v})$.

Only in case i) will the buy-out price fail to be accepted in stage one. In this case, if x denotes the second-highest valuation, the price in stage one is $b^1(x)$, and we know that *the price in stage two is at least* $b^1(x)$, since it would have been precisely $b^1(x)$ if the highest valuation had been precisely x rather than higher than x. Hence, in case i), *expected revenue in stage two exceeds the first term* in $ER_1(\widehat{v})$. We will proceed in this manner to show that, case for case, expected revenue is higher in stage two than in stage one.

Regarding the second case, consider the following. If precisely one buyer has a valuation above $\widehat{v}$ and the runner-up has a valuation x, $x \in [0, \widehat{v})$, *the price in stage two must be at least* $b^1(x)$. Again, this is because it would have been precisely $b^1(x)$, if the highest valuation had been x rather than above $\widehat{v}$, where $\widehat{v} > x$.

In the third case, let x be the valuation of the winner of stage one, which, by assumption, is between $\widehat{v}$ and $m(\widehat{v})$. Then, it is easily seen that *the term in parenthesis is the price in stage two.*

Finally, notice that in cases i), ii) and iii), the price in stage two is at most $\widehat{v}$, but in all other cases (i.e. case iv)) *the price in stage two must be at least* $\widehat{v}$.

We have now shown that, case for case, expected revenue in stage two is higher than expected revenue in stage one, which concludes the proof. ■