Pre-Auction Offers in Asymmetric First-Price and Second-Price Auctions

René Kirkegaard       Per Baltzer Overgaard
Brock University      University of Aarhus

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Abstract

We consider first-price and second-price auctions with asymmetric buyers, and examine whether pre-auction offers to a subset of buyers are profitable. A single offer is never profitable prior to a second-price auction, but may be profitable prior to a first-price auction. However, a sequence of offers is profitable in either type of auction. In our model, suitably chosen pre-auction offers work because they move the assignment when bidder valuations are “near the top” closer to the optimal, revenue-maximizing assignment.

Keywords: first-price and second-price auctions, asymmetric bidders, pre-auction offers. JEL: D02, D44, D82.

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Contact: rkirkegaard@brocku.ca and povergaard@econ.au.dk.
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1 Introduction

A seller of a unique item often faces two “problems”. On the one hand, he must sell, and, on the other, he faces heterogenous potential buyers with unknown valuations of the item. In such a setting, simply posting a price may well be counterproductive as well as non-credible, since buyers will conclude that if no one takes the posted price, some kind of negotiation or auction-like mechanism will subsequently be used by the seller to allocate the item. Also, since the seller is unable to commit not to sell the item, the revenue maximizing mechanism, in which trading occurs with a probability strictly less than one, is precluded. Hence, whatever mechanism the seller tries to set up, it must ultimately involve trading with probability one. Such mechanisms are the object of study in the present paper.\footnote{The assumption that trading must take place with probability one is not crucial. However, it serves to highlight the fact that the modification of standard auctions that we consider is profitable because it may change who wins the auction, not because it changes the probability that the object is sold. This distinction is discussed further in the next paragraph.}

In the symmetric independent private-values setting, it is well-known that any of the efficient, standard must-sell auction formats are, in fact, revenue maximizing in the class of all must-sell mechanisms (see Bulow & Klemperer (1996) and Kirkegaard (2006)).\footnote{When values are affiliated, Lopomo (1998) shows that the English auction is optimal in a large class of mechanisms, referred to as simple sequential auctions. This class of mechanisms encompasses the type of mechanisms that are the object of study in this paper. We maintain the assumption of independent private values, but we relax the symmetry assumption.} Hence, in order to raise (expected) revenue, some inefficiency must be induced through probabilistic withholding of the item.\footnote{In the symmetric setting, any standard format augmented by a suitably chosen reserve price implements the optimal transfers and trading from the perspective of seller revenue.} Thus, if withholding is ruled out, the seller cannot improve upon the standard auction formats, when buyer valuations are unknown but drawn from the same distribution. In contrast, if the symmetry assumption is dropped, it is also well-known that the standard must-sell auction formats are no longer revenue equivalent (see Maskin & Riley (2000)). In fact, under certain conditions, an inefficient first-price auction may revenue-dominate an efficient second-price format, when potential buyers are asymmetric. Since both standard auction formats are must-sell formats, the inefficiency in the first-price format is not related to withholding, but to misallocation, in the
sense that the item may not be sold to the highest-valuation bidder. Hence, this type of inefficiency may work to the advantage of the seller, and this paper investigates how the seller may try to exploit this further, when potential buyers are identifiable heterogenous \textit{ex ante}. Thus, we assume that the seller is able to identify different types of bidders, though not the actual valuation of any particular (type of) bidder.

Let us give a few of examples of what we have in mind. First, when a (local) government auctions off the rights to collect garbage or provide bus transportation in a certain area, it is often possible to identify whether particular bidders already provide similar services elsewhere or whether they are “greenfield” entrants. Also, bidders know that the contract must (ultimately) be offered to someone with probability one. Similarly, in a liquidation sale of an estate including artwork, silverware and antique furniture, the seller may be able to identify professional and private buyers. Again, all potential buyers may know that the estate must be liquidated. Finally, in a takeover contest, on the face of it there may be an obvious acquirer (e.g., a firm in a similar or complementary line of business with which the management or the board of the target firm has close ties). That is, there may be a strong potential buyer. However, there may also be a set of alternative potential acquirers, that is, weak potential buyers in our terminology. In addition, once a takeover contest has been initiated, all potential buyers may surmise that an eventual takeover is a sure thing.

In the proposed model-setting, the seller can approach buyers in sequence and make individual \textit{take-it-or-leave-it} offers. However, if all the offers are turned down, it is understood by all the parties that the item will subsequently be sold with probability one in some mechanism. Hence, we essentially introduce the possibility of making pre-auctions offers before some type of must-sell auction is staged among the asymmetric bidders. Of course, if one of the pre-auction offers is accepted, the trading mechanism never progresses to the auction stage.\footnote{Pre-auction offers and their acceptance are legally binding, and there is no default.} We study the revenue effects of such pre-auction offers when the auction stage is comprised of either a first-price auction or a second-price auction (or any other efficient mechanism).

Pre-auction offers may benefit the seller because they correct some shortcomings of the standard auctions. In particular, both kinds of auctions lead to a suboptimal assignment of the good “near the top”.\footnote{That is, if one or more bidders have valuations “near the top” of their respective...} Interestingly, the
type of misallocation from the perspective of seller revenue is very different in the first-price and second-price auctions, as we highlight in the next section. Pre-auction offers are of potential value precisely because they appeal to the “top” (bidders with high valuations), and hence may be helpful in correcting the misallocation.

We examine second-price auctions first. We show that a pre-auction offer to a single bidder is never profitable. However, when bidders are asymmetric, it is possible to construct a profitable sequence of offers, implying that several bidders will be offered the good prior to a final second-price auction. The analysis of pre-auction offers in second-price auctions is relatively straightforward since bidders use the same (weakly dominant) strategy in the auction regardless of the number of pre-auction offers.

In contrast, bidding in a first-price auction will generally depend on whether it was preceded by one or more take-it-or-leave-it offers. The reason is that if bidders are known to have rejected an offer, it is rational for their competitors to update beliefs and bidding strategies in the auction itself. As a consequence of this indirect effect of the pre-auction offer, a trade-off arises when a single offer is made prior to the first-price auction. Specifically, from the seller’s point of view, the pre-auction offer improves the assignment of the good near the top but worsens it near the bottom. In order to quantify the opposing effects we study a specific class of distributions, and show that a single pre-auction offer is indeed profitable. However, in the case where a sequence of pre-auction offers prior to a first-price auction is considered, it is possible to neutralize the problems associated with bidding strategies in the auction being contingent on the existence of pre-auction offers. More concretely, we carefully construct a sequence of pre-auction offers in which bidding strategies are not affected by the offers, and show that such a sequence is profitable regardless of the type of distributions associated with the asymmetric bidders.

The existing literature on pre-auction offers in auction-like mechanisms...
is scant. Bulow & Klemperer (1996, p. 189) remark that pre-auction offers are not profitable in the symmetric case, when a rejection of the offer is followed by a must-sell auction.\(^8\) This is, of course, immediately relevant for the takeover contest alluded to above, when potential buyers are symmetric \textit{ex ante}.\(^9\) The literature on \textit{buy-outs} in auctions is also of some relevance for this paper.\(^10\) Particularly in online auctions, sellers often stipulate a buy-out price, which will end the auction immediately, if \textit{some} bidder accepts it. This has been motivated by risk aversion or impatience on the part of either sellers or buyers and by the increasing price paths in sequential auctions associated with multi-unit demands. On the surface, buy-out offers appear similar to pre-auction offers. However, buy-out offers are general and made to all potential buyers, whereas the pre-auction offers considered here are made exclusively to some potential buyers based on \textit{ex ante} information on their types. The latter only makes sense, if the potential buyers are identifiably heterogeneous, whereas the literature on buy-outs has (so far) assumed that buyers are homogenous \textit{ex ante}.

At a more general level, this paper is related to the work of Bulow & Roberts (1989) and Bulow & Klemperer (1996), who explored the fundamental relationship between monopoly pricing and (optimal) auctions. In order to maximize profits, the monopolist will generally try to sell to buyers with the highest marginal revenues and only to those buyers whose marginal revenues exceed marginal cost. If marginal cost is taken to be the value for the seller of retaining an item for himself, then this immediately ties together optimal auction-reserves and discrimination between heterogeneous bidders with third-degree price discrimination by a monopolist. Our results similarly trade on how adaptations of standard auction formats allow the seller to “manipulate” the marginal revenue of the marginal bidder to his

\(^8\)However, Ivanova-Stenzel & Kröger (2005) suggest that pre-auction offers may raise profits in the symmetric case when bidders are \textit{risk averse}.

\(^9\)Bulow & Klemperer explicitly relate their results to U.S. takeover law. There, company boards are required to show due diligence with respect to the maximization of shareholder value before entering into \textit{exclusive negotiation} with a single potential buyer. Their main result is that effort is better spent looking for more buyers, to increase competition, rather than negotiating exclusively with one buyer. See Kirkegaard (2006) for an alternative and short proof of this result. Though related, our focus is different, in that we assume \textit{ex ante} asymmetries between potential buyers, while exclusive pre-auction offers are made on a \textit{take-it-or-leave-it} basis and always followed by a standard auction if rejected.

own advantage.

Moreover, our work is closely related to the small number of papers that study first-price auctions with asymmetric bidders, notably Griesmer, Leibman & Shubik (1967), Lebrun (1999), Maskin & Riley (2000) and Kaplan & Zamir (2006) (see also Kirkegaard (2005) and Krishna (2002, Ch. 4)). The example and the results derived on pre-auction offers in first-price auctions in this paper explicitly take the revenue rankings found by Maskin & Riley (2000) as their point of departure.

The remainder of the paper is organized as follows. Section 2 sets up the model and provides some fundamental intuitions on how and why standard auctions can be improved. In Section 3 we consider pre-auction offers in second-price auctions, while Section 4 is devoted to first-price auctions. Section 5 further discusses our results, and relate them to the literature on buy-out prices in auctions. Section 6 concludes. All proofs are in the Appendix.

2 Model and Precursor

We consider a risk neutral seller of an indivisible good, and assume the seller’s own-use value of the good is zero. There are n risk neutral buyers. Buyer i draws a private valuation independently from the distribution function $F_i$ on $[0, \bar{\tau}_i]$, $i = 1, ..., n$. $F_i$ has no mass points, is strictly increasing and continuously differentiable, with $f_i$ denoting the density. Without loss of generality, the buyers are ordered such that $\bar{\tau}_n \geq \bar{\tau}_{n-1} \geq ... \geq \bar{\tau}_1$.

Formally, we assume that the seller does not use a reserve price. However, all our results remain valid as long as any reserve is strictly below $\bar{\tau}_{n-1}$ such that at least two buyers are potentially interested in the object.\textsuperscript{11}

It will at times be convenient to derive expected revenue using Myerson’s (1981) method. Let $J_i(v) = v - \frac{1-F_i(v)}{f_i(v)}$ denote buyer i’s virtual valuation when his valuation is v.\textsuperscript{12} The (ex ante) expected payment from buyer i in

\textsuperscript{11}More precisely, the general results presented in Propositions 1, 2, and 4 are valid with a reserve price, at least as long as the pre-auction offers are not below the reserve price. In principle, if the reserve price is very high, it may be optimal to circumvent it by offering a low pre-auction offer in a second-price auction (contrary to the prediction in Proposition 1). However, a reserve price that high is not optimal in the first place.

\textsuperscript{12}For example, if $F_i(v) = v/\bar{\tau}_i$ (the uniform distribution), then $J_i(v) = 2v - \bar{\tau}_i$. 

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any particular auction or mechanism is then

\[ EP_i = \int_0^{v_i} J_i(v)q_i(v)f_i(v)dv, \]  

(1)

where \( q_i(v) \) is the equilibrium probability that buyer \( i \) wins the auction, if his valuation is \( v \).

To provide intuition for the results to follow, we will take advantage of Bu- low and Roberts’ (1989) observation that virtual valuation can be considered the counterpart to marginal revenue in the standard monopoly problem.13 This, in turn, implies that any auction can be compared to some monopoly pricing scheme. Consequently, in the following we will use Fig. 1 to illustrate the main arguments. The monopolist in Fig. 1 faces two different markets and has zero marginal costs (but potentially faces a capacity constraint). In market \( i \), willingness-to-pay ranges uniformly from 0 to \( v_i \), \( i = 1, 2 \), implying the linear (inverse) demand curves indicated. To understand the results to follow, the key is the (trivial) property that marginal revenue coincides with demand exactly at the top (or \( J_i(v_i) = \pi_i \)) but is strictly below demand everywhere else (or \( J_i(v_i) < v_i \) for \( v_i < \pi_i \)).

Fig. 1: Two asymmetric markets.

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13In all figures, we assume marginal revenue or virtual valuation is monotonic. However, this assumption is not necessary for the formal results.
from the auctioneer’s point of view. In particular, we will focus on the fact
that standard auctions fail to assign the object in the optimal way “near
the top”. That is, if all buyers have valuations close to the upper end-point
of their respective supports, standard auctions bring about a sub-optimal
allocation. In Sections 3 and 4, we will show how suitably chosen pre-
auction offers may help bring the assignment closer to the optimal assignment
near the top. Thus, pre-auction offers have the potential to improve revenue
to the seller.

Consider first a standard second-price auction. In equilibrium, buyers
submit bids equal to their valuations, which ensures that the auction is
efficient. Hence, the use of a second-price auction can be compared to
non-discriminatory pricing in a monopoly (i.e., the good is priced identically
in all markets), which also ensures that the quantity sold is allocated effi-
ciently. Thus, a second-price auction suffers from the same shortcomings as
non-discriminatory pricing from the point of view of the seller profits. In
particular, efficiency is seldomly optimal.

For instance, consider a monopolist who faces the two markets depicted in
Fig. 1, and assume total capacity is .5. Given these assumptions, it is efficient
to charge a price of 1, which effectively excludes market 1. However, at a
price of 1 it is obviously the case that marginal revenue in market 1 is strictly
higher than in market 2, implying that revenue could be improved by selling
at least part of the capacity in market 1. In Section 3, we show that a sequence
of pre-auction offers will facilitate exactly this type of discrimination. Thus,
pre-auction offers destroys the undesirable property that the second-price
auction is efficient near the top.

Turning to a first-price auction, we observe virtually the opposite prob-
lem. Specifically, the auction is too inefficient near the top. With only two
buyers in a first-price auction, there will be a common maximal bid. In
other words, if both buyers happen to realize their maximum valuations,
their bids will tie, and both buyers are equally likely to win. In Fig. 1, this
corresponds to the monopolist deciding by flipping a coin whether to sell a

\[14 \text{More precisely, this is the case if the upper end-points are not the same for all buyers. Hence, the assumption that } \pi_i \text{ and } \pi_j \text{ may differ is indispensable in our model.}

\[15 \text{This is the only equilibrium in weakly dominant strategies. Blume & Heidhues (2004) study other equilibria of the second-price auction which may be inefficient. However, they also show that the equilibrium is unique if there is a strictly positive reserve price and } n > 2.

\[16 \text{With more than 2 buyers, there may or may not be a common maximal bid.} \]
marginal quantity in market 2 at a price of 2 or in market 1 at a price of 1. Clearly, it is more profitable to sell the small quantity in market 2 with certainty. In Section 4 we show that a single pre-auction offer will allow the auctioneer to correct this undesirable feature of the first-price auction, but we also identify a potential trade-off. However, we prove that a single pre-auction offer will, on balance, increase expected revenue for a class of distributions. More generally, we show that a sequence of pre-auction offers will improve expected revenue, regardless of the distributions.

Fig. 2 provides an alternative illustration of the properties of the standard auctions for the example in Fig. 1.

![Fig. 2: Assignment of the good in second-price, first-price and optimal auctions (from “left to right”).](image)

For any given \((v_2, v_1)\)-pair, Fig. 2 shows how the object is allocated in various mechanisms. For instance, in the optimal mechanism buyer 1 wins if, and only if, his virtual valuation is higher than that of buyer 2, \(J_1(v_1) \geq J_2(v_2)\). This is the case above the fat line in Fig. 2 (buyer 1 wins), whereas the opposite is true below the line (buyer 2 wins).\(^{18}\) Fig. 2 also shows the assignment in a second-price auction. Since this auction is efficient, the (thin) 45\(^\circ\)-line captures the pivotal \((v_2, v_1)\)-pairs. Clearly, *buyer 2 wins too*

\(^{17}\)Similarly, in the auction, buyer 1 with valuation \(v_1\) will outbid buyer 2, even if buyer 2’s valuation is very close to \(v_2\). This is sub-optimal, since buyer 2’s virtual valuation exceeds \(\bar{v}_1\), if \(v_2\) is sufficiently large.

\(^{18}\)This is the optimal mechanism given the constraint that the good is sold with probability one. In the fully optimal mechanism, the good will not be sold if both buyers have negative marginal revenue.
often near the top-right corner (and, in this example, everywhere else as well), compared to what is optimal. Finally, Fig. 2 depicts the assignment in the first-price auction. The (dashed) strictly concave function describes \((v_2, v_1)\)-pairs whose bids are identical, implying that buyer 1 wins above the curve, whereas buyer 2 wins below the curve. For example, buyer 2 with valuation 1.5 bids the same as buyer 1 with valuation 0.915. Evidently, compared to the optimal mechanism buyer 1 wins too often when valuations are high (near the top-right corner).

3 Pre-Auction Offers in Second-Price Auctions

A single pre-auction offer. In the following, we first modify the second-price auction by allowing a single pre-auction offer to one of the buyers, buyer \(i\), say. That is, buyer \(i\) is offered the good at some price, \(p\). Should he decline the offer, the object is sold in a standard second-price auction (in which all buyers participate). Given that buyers play the weakly dominant strategy of bidding their true valuation in the second stage of this game, it is easy to see that buyer \(i\) must employ a threshold strategy in the first stage. That is, he accepts the offer if, and only if, his valuation is above some cut-off valuation, \(\hat{v}_i\). Regardless of which cut-off valuation \(p\) brings about, it is straightforward to calculate the probability that any given bidder wins the modified auction. Using (1), expected revenue can then easily be calculated, and its dependence on \(\hat{v}_i\) analyzed.

**Proposition 1** A single pre-auction offer followed by a second-price auction will never increase expected revenue, and may decrease it. In particular, if there is a positive probability that the offer causes an inefficient allocation, then expected revenue will strictly decrease.

**Proof.** See the Appendix. ■

To explain Proposition 1, we once again compare with monopoly pricing. Assume the monopolist depicted in Fig. 1 is deciding whether to sell the entire capacity to one particular market, rather than setting the same, capacity-clearing price in both markets (non-discriminatory pricing). The former choice allows some (extreme) discrimination between markets, whereas the latter is efficient and favors neither market. Despite this, it is
easy to see that non-discriminatory pricing dominates exclusive dealing. The reason is quite simply that by combining the two markets into one (as under non-discriminatory pricing), willingness-to-pay for a given capacity is higher than if one deals with one market only. For example, in Fig. 1, if capacity is 0.8, the choices are to sell everything in market 1 at a unit price of 0.2, to sell everything in market 2 at a unit price of 0.4, or to sell at a capacity-clearing price of 0.8 across markets. Non-discriminatory pricing is more profitable, as it yields a higher unit price.\(^{19}\)

Now, as suggested already, a second-price auction can be compared to non-discriminatory pricing, since no one is favored, and the buyer with the highest valuation wins (efficiency). Likewise, a pre-auction offer in a second-price auction is to some extent similar to exclusive dealing in a monopoly. If the buyer targeted accepts the offer, inefficiency may result because another buyer may have a higher valuation. Efficiency can be restored only by abolishing the pre-auction offer and treating all buyers the same. As in the monopoly case, the efficient auction (without a pre-auction offer) is more profitable than favoring one particular buyer.

Alternatively, consider Fig. 3 which depicts how the assignment of the good is affected by a single pre-auction offer. Panel (a) illustrates the consequences of extending an offer to buyer 2. If the offer is low enough (such that if appeals to buyer 2, even if his valuation is below \(v_1\)), buyer 2 will win even more often “at the top” than without an offer, and the assignment moves even further away from the optimal assignment.

In panel (b), the pre-auction offer is given to buyer 1. In this case, it is possible that the assignment of the good is improved from the point of view of the seller, but the opposite is also possible. On balance, however, the change is for the worse. To see this, fix any \(v_1\) for which buyer 1 would accept the offer. Then, the pre-auction offer changes the assignment only if buyer 2’s valuation is above \(v_1\), in which case buyer 2 would have won the auction with no pre-auction offer. However, contingent on this change in assignment,

\(^{19}\)An alternative explanation might be useful. Assume the monopolist is initially selling a capacity of 0.8 in market 2. By reverting to non-discriminatory pricing, and a price of 0.8, he will sell only 0.6 in market 2, and the remaining 0.2 in market 1. The loss in market 2 is captured by the lost marginal revenue on 0.2 units. Since marginal revenue is below demand, this must be below 0.8 on each unit moved from market 2 to market 1. On the other hand, the gain is on average 0.8 for each of the units moved (since revenue on each unit is 0.8). Hence, the gain exceeds the loss.
the expected virtual valuation of buyer 2 is

$$\int_{v_1}^{v_2} J_2(v_2) \frac{f_2(v_2)}{1 - F_2(v_1)} dv_2 = v_1,$$

which exceeds the virtual valuation of buyer 1, \( J_1(v_1) \). Hence, the gain from offering the pre-auction offer, \( J_1(v_1) \), does not match the expected loss, \( v_1 \). \(^{20,21}\)

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**Fig. 3.** (a) Pre-auction offer to buyer 2. (b) Pre-auction offer to buyer 1.

We should mention that Proposition 1 does not assume that virtual valuations, \( J_i(v) \), are monotonic, as is often the case in auction theory. As noted above, Bulow and Klemperer (1996) argued that pre-auction offers are not profitable in a model with symmetric buyers and monotonic virtual valuations. Hence, we generalize their result in several directions when only one pre-auction offer is allowed, but, as we show next, it is not robust if there are more pre-auction offers.

*A sequence of pre-auction offers.* Although a single pre-auction offer is unprofitable, we now show that a sequence of pre-auction offers is profitable. More specifically, we demonstrate that the auctioneer profits from offering the good first to buyer \( n \) (the buyer with the highest maximum valuation), and then to buyer \( n - 1\). If both offers are declined, a second-price auction is conducted. Once again, it is easily seen that buyer \( n \) and buyer \( n - 1 \) must both employ threshold strategies in equilibrium.

**Proposition 2** There exists a sequence of pre-auction offers which strictly increases expected revenue if \( v_n > v_{n-1} > v_{n-2} \).\(^{22}\)

\(^{20}\)This is the counterpart to the argument in footnote 19 (the monopoly interpretation).

\(^{21}\)This argument also applies to the case where the pre-auction offer is given to buyer 2. Hence, a pre-auction offer to buyer 2 is not optimal even if the optimal assignment is not globally “below” the assignment in the second-price auction (as in panel (a) of Fig. 3).

\(^{22}\)The result also holds in any second-price auction with a reserve strictly below \( v_{n-1} \).
Proof. See the Appendix.

To understand Proposition 2, consider the special case with two bidders and \( \pi_2 > \pi_1 \). Suppose the offers are such that buyer 2 accepts if, and only if, his valuation is above the maximum valuation of buyer 1, \( \hat{\pi}_2 = \pi_1 \), whereas buyer 1 accepts if, and only if, his valuation is above some \( \hat{\pi}_1 \).

If \( \hat{\pi}_1 = \pi_1 \), which is equivalent to not extending a pre-auction offer to buyer 1, the mechanism described is revenue equivalent to a second-price auction with no pre-auction offers. In either case, the good is purchased by the buyer with the highest valuation (in particular, buyer 2 wins with probability 1 if his valuation is above \( \pi_1 \)). However, it is easy to improve revenue by slightly lowering \( \hat{\pi}_1 \) below \( \pi_1 \). The reason is that buyer 1’s virtual valuation or marginal revenue is \( \pi_1 \) when his valuation is \( \pi_1 \), while buyer 2’s marginal revenue is strictly below \( \pi_1 \) when his valuation is close to \( \pi_1 \) (see Fig. 1). Hence, it is profitable to give buyer 1 preferential treatment.

A similar logic is captured by the monopoly situation in Fig. 1, if we assume that the monopolist has initially committed himself to dumping a quantity of .5 on market 2 (which is efficient). After this commitment is made, assume that capacity expands marginally for some reason. The efficient way to allocate this extra quantity would be to offer a non-discriminatory price below 1, thereby appealing to both markets. However, revenue would increase more by selling the extra capacity in market 1, where marginal revenue is high. To accomplish this, the monopolist needs to discriminate.

Fig. 4 provides another illustration of why a sequence of offers is profitable. The offer to buyer 2 does not change the assignment (at least when \( \hat{\pi}_2 = \pi_1 \)), but the offer to buyer 1 leads buyer 1 to win more often than in the standard second-price auction. Hence, the curve that separates the space into winners and losers “dips”, near \( \hat{\pi}_2 = \pi_1 \). Clearly, the overall assignment moves closer to the optimal assignment, and expected revenue therefore increases.
Corollary 1  The optimal sequence of pre-auction offers is inefficient.

Proof. If the optimal sequence is efficient, expected revenue would be the same as in a standard second-price auction, by the Revenue Equivalence Theorem. However, since the optimal sequence strictly increases expected revenue, it must be inefficient.

4 Pre-Auction Offers in First-Price Auctions

We now consider augmenting the first-price auction with a single pre-auction offer or with a sequence of pre-auction offers. We explicitly assume that $n = 2$, i.e. there are exactly two buyers. Any first-price auction with exactly two buyers has the property that the buyers share a common maximal bid. That is, buyer 1 with valuation $v_1$ submits the same bid as buyer 2 with valuation $v_2$. This property may, or may not, carry over to first-price auctions with more buyers. We also assume that $v_2 > v_1$.

As mentioned in Section 2, the common maximal bid implies that the first-price auction is too inefficient at the top compared to the revenue-maximizing mechanism. By extending a pre-auction offer to buyer 2, the auctioneer can ensure that buyer 2 wins with certainty, if his marginal revenue or virtual valuation is high.
However, pre-auction offers in first-price auctions give rise to complications that do not occur if the auction is a second-price auction. In particular, bidding strategies will depend on beliefs in a first-price auction, but not in a second-price auction (where the strategy is weakly dominant). Clearly, should the auction materialize, it must be because buyer 2 does not have a sufficiently high valuation to justify accepting the offer. Given this update, buyer 1 has less of an incentive to bid aggressively in the auction, if the auction is a first-price auction.

A single pre-auction offer. The existence of the trade-off makes it difficult to say whether a single pre-auction offer is profitable. In the following, we discuss an example in some detail, and show that a single pre-auction offer may strictly increase expected revenue. Hence, assume that both buyers draw valuations from uniform distributions, stretched over different intervals. In particular, we normalize $v_1 = 1$, and assume that $v_2 > 1$. When $v_2 = 2$, Fig. 1 illustrates the demand, and Fig. 5 describes equilibrium bidding strategies in a first-price auction with no pre-auction offer.\(^{23}\) Since buyer 1 is a “weaker” buyer (being less likely to have a high valuation) than buyer 2, he submits higher bids.

![Fig. 5: Equilibrium bidding strategies](image)

The effect on equilibrium bidding behavior of a pre-auction offer to buyer 2 is illustrated in Fig. 6 (fat curves refer to the case with a pre-auction offer, while thin curves replicate Fig. 5).

With a pre-auction offer, we note that buyer 1 - the weak bidder - bids less aggressively in the auction when the strongest opponent types have been eliminated by accepting the offer. This captures that, for the weak bidder, competition has become weaker, and it is expected to take less to win. In contrast, the remaining, “low” types of buyer 2 - the strong bidder - bid more aggressively. To see this, recall that in a first-price format, where the winner pays his bid, any bidder must weigh the decrease in payoff from winning against the increased probability of winning when the bid is raised. But here the increased density of opponent bids (due to the compression of buyer 1’s bidding interval) tilts this cost-benefit trade-off in favor of higher bids. Hence, the remaining types of the strong buyer bid more aggressively.

Finally, Fig. 7 illustrates the consequences of a pre-auction offer to buyer 2 on the assignment of the item. The concave function which identifies who wins and who loses moves left as a consequence of the offer. The reason is that buyer 1 bids relatively less aggressively, buyer 2 bids more aggressively, and buyer 2 wins for sure near the top. This, in turn, implies that the assignment moves closer to the optimal assignment for high valuations, but further away from the optimal assignment for low valuations. Hence, the trade-off.
For the uniform case, we can state the following result.

**Proposition 3 (The Uniform Case)** If \( F_1(v) = v \) and \( F_2(v) = \frac{v}{\overline{v}_2} \), \( \overline{v}_2 > 1 \), there exists a pre-auction offer to buyer 2 which strictly increases expected revenue.\(^{24}\)

**Proof.** See the Appendix. \( \blacksquare \)

In the case of a single pre-auction offer, we notice that efficiency is improved. In Fig. 7, the assignment moves closer to the 45° line, which represents an efficient mechanism. The offer not only improves efficiency at the top, as it is designed to do, it also improves efficiency at the bottom by making buyer 1’s bid closer to buyer 2’s bid.

**Corollary 2** If \( F_1(v) = v \) and \( F_2(v) = \frac{v}{\overline{v}_2} \), \( \overline{v}_2 > 1 \), the revenue maximizing pre-auction offer increases efficiency.

**Proof.** It is easily proven that the assignment moves left-ward in Fig. 7 when buyer 2’s threshold for acceptance, \( \widehat{v}_i \), is lowered.\(^{25}\) Thus, the corollary follows

\(^{24}\)In addition to the fact that a pre-auction offer improves revenue, it is interesting to observe that the threshold in the example with \( \overline{v}_2 = 2 \) should be strictly higher than 1.5, the point at which buyer 2’s marginal revenue (virtual valuation) enters the range of buyer 1’s marginal revenue from above. The reason is that by reducing the asymmetry too much in the auction, the assignment near the bottom moves too far away from what is optimal.

\(^{25}\)Plum (1992) finds that the two buyers will submit the same bid in the first price auction if \( v_2 = v_1(1 - cv_1^2)^{-1/2} \), where \( c = 1 - \overline{v}^{-2} \) (equation (4.10) in Plum (1992), with \( \overline{v}_1 = 1, \overline{v}_2 = \widehat{v} \)). For given \( v_1, v_2 \) must clearly decrease if \( \widehat{v} \) decreases.
by proving that the optimal cut-off is not below $\bar{v}_1 = 1$.\textsuperscript{26} By contradiction, if $\hat{v} < \bar{v}_1$, the assignment is clearly far removed from the optimal assignment (it would be to the left of the $45^\circ$ line), and there would be no trade-off from moving it right-ward, which can be accomplished by increasing the cut-off (or the pre-auction offer).

A sequence of pre-auction offers. We next consider a sequence of pre-auction offers. Specifically, we assume buyer 2 is given an offer first. If he rejects, an offer is extended to buyer 1. Should he also reject, a standard first-price auction is conducted. As before, it is easily seen that both buyers employ threshold strategies. We consider general distribution functions, but maintain the assumption that $\bar{v}_2 > \bar{v}_1$.

In the following, we construct a sequence of offers which improves expected revenue. As is the case with a single pre-auction offer, a sequence of pre-auction offers alters beliefs if the offers are rejected. Therefore, bidding behavior in the auction may be affected. However, we carefully select the cut-off valuations (by selecting the offers) in such a manner that auction behavior is not affected. Consequently, the assignment for low valuations will be unchanged in our construction.

We start by examining bidding in a first-price auction with no pre-auction offers. Buyer 2’s bid is the best response to buyer 1’s bidding strategy, $b_1$, given beliefs about buyer 1, which are summarized by $F_1$. Alternatively, we can think of buyer 2’s problem in terms of picking a critical value, $\tau$, such that he outbids buyer 1 if buyer 1 has valuation below $\tau$, but fails to match buyer 1 if buyer 1 has valuation above $\tau$. In other words, his bid would tie with that of buyer 1, if buyer 1 has valuation $\tau$. If buyer 2 has valuation $v$, his problem can then be written as

$$\max_{\tau} (v - b_1(\tau))F_1(\tau),$$

where $\tau \in [0, \bar{v}_1]$. The strictly concave function in Fig. 2 (and the right-most in Fig. 7) is $\tau(v)$ in the uniform example. In equilibrium, both buyers’ bids are best responses, implying that buyer 2 with valuation $v$ picks the critical value $\tau$, while buyer 1 with valuation $\tau$ picks the critical value $v$, $\tau \in [0, \bar{v}_1]$, $v \in [0, \bar{v}_2]$. Importantly, both buyers share the same maximal

\textsuperscript{26}The mechanism is efficient (the buyer with the highest valuation wins) if buyer 2’s cut-off is $\hat{v} = \bar{v}_1 = 1$. If the cut-off is lower, efficiency starts decreasing again, since in this case buyer 2 will start winning more often than is efficient.
bid, \( b_1(\bar{v}_1) = b_2(\bar{v}_2) \), or \( \tau(\bar{v}_2) = \bar{v}_1 \). If this was not the case, the buyer with the highest possible bid could benefit from reducing his bid slightly, since it would not change the probability that he wins.

Next, consider a pre-auction offer to buyer 1, say. Should buyer 1 reject the offer, then buyer 2 infers that \( v_1 \) is smaller than the cut-off, \( \bar{v}_1 \). However, buyer 2’s problem takes the same form as before,

\[
\max_{\tau'} (v - b_1(\tau')) \frac{F_1(\tau')}{\bar{v}_1}. 
\]

(3)

Assuming for the moment that buyer 1 continues to use the same strategy in the auction as without a pre-auction offer, the problem for buyer 2 is essentially the same, since \( 1/F_1(\bar{v}_1) \) is just an irrelevant constant. The only, though important, difference is that the range of \( \tau \) has changed from \([0, \bar{v}_1]\) to \([0, \bar{v}_1]\), implying that the highest possible bid submitted by buyer 1 in the auction is lowered. If buyer 2’s valuation is low, his best response (and his bid) would be the same as before. However, if his valuation is high (such that \( \tau(v) > \bar{v}_1 \) initially) his optimal bid changes. It is suboptimal to submit a bid strictly higher than what buyer 1 would ever submit. Since buyer 2’s bidding strategy changes, buyer 1’s best response also changes, and as a result the new equilibrium strategies may be significantly different from the equilibrium strategies in the game with no pre-auction offer. This is what lead to the trade-off in the auction with a single pre-auction offer.

Now, consider the following sequence of offers. First, buyer 2 is given an offer, which is designed in such a manner that he accepts if, and only if, his valuation is above some \( \hat{v} \). If he rejects, buyer 1 is given an offer which appeals to him if, and only if, his valuation is above \( \tau(\hat{v}) \), where \( \tau \) is from the original game with no offers. The advantage of this pair of offers, or cut-offs, is that the original strategies remain best responses to each other. Assuming buyer 1 does not change his bidding strategy, the function \( \tau(v) \) still solves the problem, when \( v \in [0, \hat{v}] \). Contrary to the case with a single pre-auction offer, this particular sequence of offers, by design, “cuts off” or truncates the bids the same place, such that the highest maximal bid is still the same on the new supports, \([0, \tau(\hat{v})]\) and \([0, \hat{v}]\), respectively. Since bidding is the same in the auction, the assignment is unaffected by the pre-auction offers, if both buyers have low valuations. Thus, there is no trade-off.

For instance, in the example in Fig. 1 and Fig. 2, if \( \hat{v} = 1.5 \) then \( \tau(\hat{v}) \approx .915 \), and the common maximal bid after the sequence of offers is approximately .568. Fig. 8 overlays the resulting assignment on top of Fig.
2. Obviously, the assignment has moved closer to the optimal assignment, compared to the auction with no pre-auction offers. This argument holds for any distributions on different supports, at least when \( \hat{v} \) is chosen to satisfy \( J_2(\hat{v}) = J_1(\overline{v}_1) = \overline{v}_1 \) (or the highest such value if there are more, which could occur if \( J_2 \) is non-monotonic). Thus, we conclude that the sequence of offers increases expected revenue.

![Figure 8: A sequence of offers prior to a first-price auction.](image)

To implement the cut-offs \( \hat{v} \) and \( \tau(\hat{v}) \), the offer to buyer 2, \( p_2 \), and to buyer 1, \( p_1 \), solves

\[
\begin{align*}
\hat{v} - p_2 &= F_1(\tau(\hat{v}))(\hat{v} - b_2(\hat{v})) \\
\tau(\hat{v}) - p_1 &= \tau(\hat{v}) - b_2(\hat{v}),
\end{align*}
\]

which ensure that the buyers are indifferent between accepting and rejecting the offers if their valuations are exactly the cut-off valuations.\(^{27}\) If the offer is rejected by such a buyer, it is easily seen that the optimal strategy in the first-price auction would be to bid \( b_2(\hat{v}) \), the common maximal bid, thereby guaranteeing that the buyer wins.

We end by stating our main result for pre-auctions offers in the first-price setting.

**Proposition 4** There exists a sequence of pre-auction offers which strictly increases expected revenue if \( \overline{v}_2 > \overline{v}_1 \).

**Proof.** See the Appendix. \( \blacksquare \)

\(^{27}\)Notice that \( p_1 = b_2(\hat{v}) = b_1(\tau(\hat{v})) \). It is straightforward to show that buyers with valuations different from the cut-off find it optimal to comply with the threshold strategy.
5 Temporary buy-out prices with sequential arrival

Consider an auction with a temporary buy-out price. The buy-out price represents a simultaneous offer to all buyers. That is, the first buyer to accept the buy-out price wins the auction. The buy-out price is temporary, as on eBay, meaning that as soon as a bid is submitted in the auction the buy-out price disappears.

We assume that buyers arrive at the auction in some (possibly random) sequence, with zero probability of two buyers arriving at exactly the same time. If the auction is a second-price, sealed-bid auction or an English auction, it is optimal for the first buyer to arrive to either accept the buy-out price or to submit a bid which will nullify the buy-out price and thereby erase the option of accepting the buy-out price for other buyers. In effect, the buy-out price is only available to the first bidder. Proposition 1 then implies that the seller is worse off.

**Corollary 3** In a second-price auction with sequential arrival a temporary buy-out price will never increase expected revenue, and may decrease it.

Corollary 3 suggests that the existence of buy-out prices in online auctions cannot be explained by asymmetries among buyers. However, the corollary does not apply if the buy-out price is permanent, as is the case on Yahoo!.

Moreover, Corollary 3 also assumes that auctions with buy-out prices are best described as second-price auctions. However, this is not necessarily the case. If the auction is of an open ascending format (like the English auction) but has a firm closing time, there might be an incentive for buyers to engage in sniping. That is, instead of bidding early (and potentially being outbid), a bid that is submitted just seconds before the closing of the auction might not give other buyers time to respond. In this case, the winner pays his own bid, implying the auction has a first-price feel to it. Proposition 3 suggests that the seller might potentially benefit from a temporary buy-out price in this case.

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28 In Proposition 2 the offers potentially depend on the identity of the buyer in question. Hence, Proposition 2 does not carry over to the case of permanent buy-out prices (where the offer is the same to all buyers).
6 Concluding Remarks

Standard, must-sell auctions can be improved from the seller’s perspective by the introduction of targeted pre-auction offers when potential buyers are identifiably heterogenous. A single pre-auction offer is counter-productive for the seller, when the auction is of the second-price format, but a well-chosen sequence of offers raises expected revenue. In the case of first-price auctions, sometimes even a single pre-auction offer raises expected revenue, while a well-chosen sequence certainly does. These results trade on the ability of pre-auction offers to affect the assignment of the item for bidder valuations near the top of their respective ranges.
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Appendix

Proof of Proposition 1. We consider the possibility that the seller makes a pre-auction offer to some buyer, buyer $i$, say. Buyer $i$ accepts if his valuation is at least $\tilde{v}$. Notice that if $i = n$ and $\tilde{v} \geq \overline{v}_{n-1}$ with $\overline{v}_n > \overline{v}_{n-1}$, the pre-auction offer does not change the allocation, as buyer $n$ would win regardless, when his valuation exceeds $\overline{v}_{n-1}$. By the Revenue Equivalence Theorem, revenue is unaffected. Hence, we consider thresholds below $\overline{v}_{n-1}$ in the following.

Let $A$ be the set of buyers other than buyer $i$ who are affected by the pre-auction offer. If $j \in A$, then $\overline{v}_j \geq \tilde{v}$, meaning that there is a chance buyer $j$ has the highest valuation, yet loses to buyer $i$. Let $B$ be the set of buyers not in $A$ (with $\overline{v}_j < \tilde{v}$) and different from $i$. Finally, let $G_j(v) = \prod_{k \neq j} F_k(v)$, be the probability that buyer $j$ with valuation $v$ has the highest valuation.

In the following, we will use Myerson’s (1981) method of writing expected revenue. Hence, we can write expected revenue as

\[
ER_i(\tilde{v}) = \sum_{j \in A \cup \{i\}} \int_0^{\overline{v}_j} J_j(v)G_j(v)f_j(v)dv + \sum_{j \in B} \int_0^{\overline{v}_j} J_j(v)G_j(v)f_j(v)dv \\
+ \int_{\tilde{v}}^{\overline{v}_i} J_i(v)f_i(v)dv + \sum_{j \in A} \int_{\tilde{v}}^{\overline{v}_j} J_j(v)F_i(\tilde{v}) \prod_{k \neq j,i} F_k(v)f_j(v)dv,
\]

since buyer $j$ with a valuation below $\tilde{v}$ wins if he has the highest valuation, which occurs with probability $G_j(v)$, buyer $i$, to whom the offer is made, wins with probability one if his valuation is above $\tilde{v}$, and buyer $j \neq i$ with valuation above $\tilde{v}$ wins if he has the highest valuation and buyer $i$ has a valuation below $\tilde{v}$.

In contrast, expected revenue in a second-price auction without a pre-auction offer is

\[
ER_{SPA} = \sum_{j=1}^{n} \int_0^{\overline{v}_j} J_j(v)G_j(v)f_j(v)dv.
\]
Revenue from the two auctions can now be compared,

\[
D_i(\hat{v}) \equiv ER_{SPA} - ER_i(\hat{v}) = \sum_{j \in A \cup \{i\}} \int_{\hat{v}}^{\pi_j} J_j(v) G_j(v) f_j(v)dv \\
- \left[ \int_{\hat{v}}^{\pi_i} J_i(v) f_i(v)dv + \sum_{j \in A} \int_{\hat{v}}^{\pi_j} J_j(v) F_i(\hat{v}) \prod_{k \neq j,i} F_k(v) f_j(v)dv \right].
\]

The first term is identical to expected revenue in a second-price auction with reserve price of \(\hat{v}\) (which buyers in \(B\) never win). The second term is revenue in a mechanism where buyer \(i\) is offered the good at a price of \(\hat{v}\), and if he rejects, all other buyers are invited to a second-price auction with a reserve price of \(\hat{v}\). Clearly, the former auction is more profitable, implying that \(D_i(\hat{v}) > 0\) as we sought to prove.

An alternative proof starts by examining the derivative of \(ER_i(\hat{v})\),

\[
ER_i'(\hat{v}) = f_i(\hat{v}) \left[ \sum_{j \in A} \int_{\hat{v}}^{\pi_j} J_j(v) \prod_{k \neq j,i} F_k(v) f_j(v)dv - J_i(\hat{v})(1 - G_i(\hat{v})) \right].
\]

The first term in brackets is equivalent to the expected revenue in a second-price auction with a reserve price of \(\hat{v}\) among all the buyers except buyer \(i\). This clearly exceeds revenue from posting a price of \(\hat{v}\), which would yield expected revenue of \(\hat{v}(1 - G_i(\hat{v}))\). Hence,

\[
ER_i'(\hat{v}) \geq f_i(\hat{v})(1 - G_i(\hat{v}))(\hat{v} - J_i(\hat{v})) = (1 - G_i(\hat{v}))(1 - F_i(\hat{v})) \geq 0.
\]

It follows that a single pre-auction offer never improves revenue in a second-price auction. Notice that we have not assumed that \(J_j\) is monotonic. Bulow and Klemperer (1996) argued that pre-auction offers are not profitable in a model with symmetric buyers and monotonic virtual valuations. Hence, we generalize this result in several directions.

To reveal the intuition behind the result we rearrange the derivative,

\[
ER_i'(\hat{v}) = f_i(\hat{v})(1 - G_i(\hat{v})) \left[ \sum_{j \in A} \int_{\hat{v}}^{\pi_j} J_j(v) \prod_{k \neq j,i} F_k(v) \frac{f_j(v)}{(1 - G_i(\hat{v}))}dv - J_i(\hat{v}) \right].
\]

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29 Clearly, for any \(i\), \(A\) and \(B\) depend on \(\hat{v}\). Given \(A\) and \(B\), however, revenue increases by increasing \(\hat{v}\) until buyers move from \(A\) to \(B\) (\(\hat{v} = \min_{j \in A} \pi_j\)). As we reduce \(A\) further, revenue increases. For any given \(i\), it follows that the optimal \(\hat{v} = \pi_i\), which is equivalent to no pre-auction offer.
If the pre-auction offer changes the allocation, it is because buyer $i$ wins when another buyer (in $A$) has a higher valuation. In this event, the gain contributing to an increase in revenue is the virtual valuation of buyer $i$, $J_i(\hat{v}) \leq \hat{v}$. The loss, however, is the virtual valuation of a buyer known to have a higher valuation. This is at least equal to $\hat{v}$. On a market where the lowest willingness-to-pay is $\hat{v}$, a monopolist can get at least $\hat{v}$ for each of his units (and strictly more if he faces a capacity constraint). So, the average marginal revenue, or virtual valuation, is at least $\hat{v}$. Hence, the loss from introducing a pre-auction offer exceeds the gain.

**Proof of Proposition 2.** Assume that $\hat{v}_n = \tau_{n-1}$ and that $\hat{v}_{n-1} \in [\tau_{n-2}, \tau_{n-1}]$. Notice that if $\hat{v}_{n-1} = \tau_{n-1}$, the allocation in the mechanism is the same as in a standard second-price auction (in particular, buyer $n$ wins with certainty whenever his valuation exceeds $\tau_{n-1}$). We will show that expected revenue strictly increases if $\hat{v}_{n-1}$ is lowered marginally, implying there is a sequence of offers which strictly increases expected revenue. Using Myerson’s (1981) method, expected revenue as a function of $\hat{v}_{n-1}$ is

$$ER(\hat{v}_{n-1}) = \sum_{j=1}^{n-2} \int_0^{\tau_j} J_j(v)G_j(v)f_j(v)dv + \int_{\tau_{n-1}}^{\hat{v}_{n-1}} J_{n-1}(v)G_{n-1}(v)f_{n-1}(v)dv$$

$$+ \int_{\tau_{n-1}}^{\tau_{n-1}} J_{n-1}(v)F_n(\tau_{n-1})f_{n-1}(v)dv + \int_{0}^{\hat{v}_{n-1}} J_n(v)G_n(v)f_n(v)dv$$

$$+ \int_{\tau_{n-1}}^{\hat{v}_{n-1}} J_n(v)F_{n-1}(\hat{v}_{n-1})f_n(v)dv + \int_{\tau_{n-1}}^{\tau_n} J_n(v)f_n(v)dv, \quad (4)$$

where $G_j(v)$ is defined as in the proof of Proposition 1. To understand (4), notice that any buyer with valuation below $\hat{v}_{n-1}$ wins only if he outbids his rivals in the auction (he is either not given a pre-auction offer, or he is not interested in accepting the pre-auction offer). Buyer $n-1$ with a valuation above $\hat{v}_{n-1}$ accepts the offer (and wins) if the offer is extended to him, which happens only if buyer $n$ has valuation below $\tau_{n-1}$ (thus rejecting the first offer). Buyer $n$ with valuation above $\tau_{n-1}$ accepts the offer, and thus wins with probability one. Finally, if buyer $n$ has a valuation between $\hat{v}_{n-1}$ and $\tau_{n-1}$ he will reject the offer, but win the second price auction if it materializes. This happens only if buyer $n-1$ rejects the offer given to him, which occurs with probability $F_{n-1}(\hat{v}_{n-1})$. To proceed, we use Leibniz’s rule to find the
derivative of $ER(\hat{v}_{n-1})$,

$$ER'(\hat{v}_{n-1}) = f_n(\hat{v}_{n-1}) \int_{\hat{v}_{n-1}}^{\tau_{n-1}} (J_n(v) - J_{n-1}(\hat{v}_{n-1})) f_n(v) dv.$$ 

Although $ER'(\tau_{n-1}) = 0$, it is easily seen that $ER'(\hat{v}_{n-1})$ is strictly negative if $\hat{v}_{n-1}$ is slightly below $\tau_{n-1}$. This follows from the fact that $J_{n-1}(\hat{v}_{n-1}) \approx \tau_{n-1}$ when $\hat{v}_{n-1}$ is close to $\tau_{n-1}$, while $J_n(v)$ is strictly below $\tau_{n-1}$ for $v \leq \tau_{n-1}$ (see Fig. 1). More formally, $ER''(\tau_{n-1}) > 0$, implying that $ER'(\hat{v}_{n-1}) < 0$ close to $\tau_{n-1}$ since $ER'(\tau_{n-1}) = 0$. Consequently, the optimal value of $\hat{v}_{n-1}$ is strictly below $\tau_{n-1}$. 

**Proof of Proposition 3.** Assume buyer 1’s valuation is drawn from the uniform distribution over [0, 1], and, for now, that buyer 2’s valuation is drawn from the uniform distribution over [0, $b_{v}$], $b_{v} \geq 1$. Then, we know from the analysis of Plum (1992), Maskin and Riley (2000) or Krishna (2002) that the highest bid is

$$\overline{b}(\overline{v}) = \frac{\overline{v}}{1 + \overline{v}},$$

while the inverse bid functions are $v_1(b) = \frac{2b}{1 + (1 - \frac{1}{b^2})b^2}$ and $v_2(b) = \frac{2b}{1 + (\frac{1}{b^2})b^2}$. Then, the probability that the winning bid is below $b$, $H(b; \overline{v})$, is the probability that both buyers have valuations below the valuation for which a bid of $b$ would be submitted,

$$H(b; \overline{v}) = \frac{2b}{1 + (1 - \frac{1}{b^2})b^2} \times \frac{1}{\overline{v}} \left( \frac{2b}{1 + \left( \frac{1}{b^2} - 1 \right)b^2} \right) = \frac{4b^2}{\overline{v}} \left( 1 + b^2 \frac{2\overline{v}^2 - \overline{v}^4 - 1}{\overline{v}^4} \right)^{-1}. \quad (6)$$

Letting $h(b; \overline{v})$ be the density of the winning bid, expected revenue in a first-price auction with the distributions under consideration would therefore be

$$\int_0^{\overline{v}} bh(b; \overline{v}) db = \overline{b}(\overline{v}) - \int_0^{\overline{v}} H(b; \overline{v}) db, \quad (7)$$

which can be computed, given (6).

We now turn to the first-price auction with a pre-auction offer. We let $\tau_2$ denote the highest possible valuation of buyer 2, the strong buyer. Letting $p$ denote the pre-auction offer, we claim the strong buyer accepts $p$ if, and only if, his valuation exceeds $\hat{v}$, where $\hat{v}$ (uniquely) solves $p = \overline{b}(\overline{v}) = \frac{\overline{v}}{1 + \overline{v}}$. We start
by assuming the proposed strategies form an equilibrium, and confirm this by showing that there is no incentive to deviate. First, if the auction stage is reached, the beliefs, given the strategy in the first stage, is that buyers’ valuations are drawn from uniform distributions over \([0, 1]\) and \([0, \hat{v}]\), respectively. Given this, the equilibrium bidding strategies are outlined above. If the strong buyer deviates in the first stage, rejecting \(p\) when his valuation was above \(\hat{v}\), it is easy to show that his best response in the first price auction is to submit the highest bid, \(\frac{\hat{v}}{1+\hat{v}}\). Thus, regardless of whether he accepts or rejects, he will win with probability one, and he will pay \(\frac{\hat{v}}{1+\hat{v}}\). Hence, there is no incentive to deviate. Neither is there an incentive for the strong buyer with a valuation below \(\hat{v}\) to accept \(p\). The reason is that he can just submit a bid of \(\frac{\hat{v}}{1+\hat{v}}\) in the second stage and win with probability one, so he will be no worse off rejecting.

In the mechanism with a pre-auction offer, the good may be sold in the first stage, at the price stipulated by the auctioneer, \(\frac{\hat{v}}{1+\hat{v}}\), or it may be sold in the second stage, where bidding strategies have already been outlined, given general beliefs on \(\hat{v}\). Specifically, the object is sold in the first round with probability \(\frac{\pi_2 - \hat{v}}{\pi_2}\), i.e. the probability that the strong buyer has a valuation in excess of \(\hat{v}\). With probability \(\frac{\hat{v}}{\pi_2}\), the strong buyer rejects the offer, in which case the weak buyer updates his beliefs, and the expected revenue will therefore be \((7)\). Hence, expected revenue in the new mechanism is, as a function of \(\hat{v}\),

\[
ER(\hat{v}) = \frac{\pi_2 - \hat{v}}{\pi_2} \bar{b}(\hat{v}) + \frac{\hat{v}}{\pi_2} \left( \bar{b}(\hat{v}) - \int_0^{\bar{b}(\hat{v})} H(b; \hat{v})db \right) = \bar{b}(\hat{v}) - \frac{\hat{v}}{\pi_2} \int_0^{\bar{b}(\hat{v})} H(b; \hat{v})db.
\]

Using Leibniz’s rule, the derivative is

\[
ER'(\hat{v}) = \bar{b}'(\hat{v}) - \frac{1}{\pi_2} \int_0^{\bar{b}(\hat{v})} H(b; \hat{v})db - \frac{\hat{v}}{\pi_2} \bar{b}'(\hat{v}) - \frac{\hat{v}}{\pi_2} \int_0^{\bar{b}(\hat{v})} \frac{\partial H(b; \hat{v})}{\partial \hat{v}}db,
\]

since \(H(\bar{b}(\hat{v}); \hat{v}) = 1\). In particular, at \(\hat{v} = \pi_2\) the derivative is

\[
ER'(\pi_2) = \int_0^{\pi_2} \left( \left( -\frac{\partial H(b; \hat{v})}{\partial \hat{v}} \right)_{\hat{v} = \pi_2} - \frac{H(b; \hat{v})}{\pi_2} \right) db.
\]

\(^{30}\)To see this, notice that the strong buyer with valuation \(v\) tries to maximize \((v-b)q_*(b)\), where \(q_*(b)\) is the probability that the weak buyer bids below \(b\). At any \(b\) where the first derivative is zero for a \(v\) below \(\hat{v}\), it must be strictly positive for \(v > \hat{v}\).
Notice that $\frac{\partial H(b;\bar{v})}{\partial v}$ is negative, since it becomes less likely that the winning bid is small when the strong bidder grows stronger. However, we can evaluate

\[
\left(-\frac{\partial H(b;\bar{v})}{\partial v}|_{\bar{v}=\bar{\tau}_2}\right) - \frac{H(b;\bar{\tau}_s)}{\bar{\tau}_2} = \frac{4b^\beta}{\bar{\tau}_2}\left(1 + b^\beta \frac{2\bar{\tau}_2 - \bar{\tau}_1 - 1}{\bar{\tau}_2}\right)^{-2}
\]

\[
\times \frac{(4\bar{\tau}_2 - 4\bar{\tau}_3)\bar{\tau}_1^2 - 4\bar{\tau}_3^2(2\bar{\tau}_2 - \bar{\tau}_1 - 1)}{\bar{\tau}_2^4},
\]

which is strictly negative when $b > 0$ and $\bar{\tau}_2 > 1$ (and zero if either $b = 0$ or $\bar{\tau}_2 = 1$). Consequently, $ER'(\bar{\tau}_2) < 0$, and it follows that a pre-auction offer ($\bar{v} < \bar{\tau}_2$) improves expected revenue. ■

**Proof of Proposition 4.** Using Myerson’s (1981) method, expected revenue, as a function of $\bar{v}$ is

\[
ER(\bar{v}) = \int_0^{\tau(\bar{v})} J_1(v)q_1(v)f_1(v)dv + \int_{\tau(\bar{v})}^{\bar{\tau}_1} J_1(v)F_2(\bar{v})f_1(v)dv
\]

\[
+ \int_0^{\bar{\tau}_2} J_2(v)q_2(v)f_2(v)dv + \int_{\bar{\tau}_2}^{\bar{\tau}_1} J_2(v)f_2(v)dv,
\]

where $q_i(v)$ is the equilibrium probability that buyer $i$ with valuation $v$ wins the object. As mentioned, the cut-offs $\bar{v}$ and $\tau(\bar{v})$, respectively, are chosen such that bidding strategies do not change in the first-price auction, meaning that $q_i(v)$ is independent of $\bar{v}$. The derivative is

\[
ER'(\bar{v}) = f_2(\bar{v})\int_{\tau(\bar{v})}^{\bar{\tau}_1} J_1(v)f_1(v)dv - f_2(\bar{v})J_2(\bar{v})(1 - q_2(\bar{v}))
\]

since $q_1(\tau(\bar{v})) = F_2(\bar{v})$, by definition of $\tau(\cdot)$. Moreover, since $q_2(\bar{v}) = F_1(\tau(\bar{v}))$, again by definition, we may write

\[
ER'(\bar{v}) = f_2(\bar{v})\int_{\tau(\bar{v})}^{\bar{\tau}_1} (J_1(v) - J_2(\bar{v}))f_1(v)dv.
\]

Although $ER'(\bar{\tau}_2) = 0$ (since $\tau(\bar{\tau}_2) = \bar{\tau}_1$), it is easily seen that $ER'(\bar{v})$ is strictly negative if $\bar{v}$ is slightly below $\bar{\tau}_2$. This follows from the fact that when $\bar{v} \approx \bar{\tau}_2$, then $J_2(\bar{v}) \approx \bar{\tau}_2$, which exceeds any value $J_1(v)$ can take. More formally, $ER''(\bar{\tau}_2) > 0$, implying that $ER'(\bar{v}) < 0$ close to $\bar{\tau}_2$ since $ER'(\bar{\tau}_2) = 0$. ■