Optimal Housing, Consumption, and Investment Decisions
over the Life-Cycle\textsuperscript{a}

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ABSTRACT: We provide explicit solutions to life-cycle utility maximization problems involving dynamic decisions on investments in stocks and bonds, consumption of perishable goods, and the rental and the ownership of residential real estate. House prices, stock prices, interest rates, and the labor income of the decision-maker follow correlated stochastic processes. The explicit consumption and investment strategies are simple and intuitive and are thoroughly discussed and illustrated in the paper. A calibrated version of the model exhibits many interesting and realistic features. For example, due to a positive correlation between house prices and labor income, young individuals want little (or even negative) exposure to house price risk and tend to rent their home. Later in life, when human wealth declines, the desired housing investment increases and will eventually reach and exceed the desired housing consumption, suggesting that the individual should buy his home—and buy either additional housing units (for renting out) or house price linked financial assets. In the final years, preferences shift back to home rental. While our model involves continuous adjustments of the consumption of housing services and the exposure of wealth to house price risk, we have demonstrated that the derived strategies are still very useful if the housing positions are only reset infrequently. Our results suggest that markets for REITs or other financial contracts linked to house prices will lead to non-negligible improvements of welfare.

KEYWORDS: Housing, labor income, portfolio choice, life-cycle decisions, REITs

JEL-Classification: G11, D14, D91, C6
1 Introduction

The two largest assets for many individuals are the human capital and the residential property owned and occupied by the individual. The financial decisions of individuals over the life-cycle are bound to be affected by the characteristics of these assets. While the early literature on dynamic consumption and portfolio decisions (Samuelson 1969; Merton 1969, 1971) ignored such non-financial assets, a number of recent papers have included either labor income or housing aspects in a life-cycle framework of consumption and portfolio choice (references are given below). The few papers incorporating both labor income and housing decisions impose a strict relation between the consumption of housing services and the investment exposure to house price risk, and they resort to computationally intensive numerical solution techniques. In contrast, this paper allows more flexibility in housing consumption and investment decisions and provides closed-form solutions for continuous-time problems involving both consumption, housing, and investment decisions when stock prices, interest rates, labor income, and house prices vary stochastically over time. The closed-form solutions lead to a deeper understanding of the economic forces driving individual decisions in such a complex setting. A calibrated version of our model generates life-cycle patterns in consumption and investments with many interesting and realistic features.

Our model has the following features. The individual has a time-additive Cobb-Douglas type utility function of consumption of a perishable good and of housing services. The individual receives an exogenous stochastic stream of labor income until a fixed retirement date after which the individual lives for another fixed period of time earning a constant fraction of the income level prior to retirement. Our specification of the income process encompasses life-cycle variations in the expected growth rate and volatility and also allows for variations in expected income growth related to the short-term interest rate in order to reflect dependence on the business cycle. The pure financial assets available are a stock, a bond, and short-term deposits (cash). The short-term interest rate and the returns on the bond are modeled by the Vasicek model, and for the stock price we assume a constant expected excess return, a constant volatility, and a constant correlation with the bond price. The individual can buy and sell houses\(^1\) at a unit price that varies stochastically with a constant expected growth rate in excess of the short-term interest rate, a constant volatility, and constant correlations with labor income and financial asset prices. The purchase of a house serves a dual role by both generating consumption services and by constituting an investment affecting future wealth and consumption opportunities. We allow the individual to disentangle the two dimensions of housing by renting the house instead of owning it (the rent is assumed proportional to the price of the house rented). Alternatively, investments in either shares in REITs (Real Estate Investment Trusts) or the S&P/Case-Shiller Home Price Indices (CSI) futures and options allow individuals to disconnect the physical ownership and occupancy of houses from the financial exposure to house prices.\(^2\)

\(^1\)In order to keep the terminology simple we use “house” instead of the more general term “residential property.”

\(^2\)A REIT is an investment company that invests in (and often operates) real estate generating rental income and
Our closed-form solution allows us to break down the optimal investment strategy into speculative demands, hedging demands, and so-called income-adjustment terms. In unconstrained models with labor income, it is well-known that any investor will seek a certain exposure of his total wealth to the different risks, depending on the risk aversion and hedging motives; see, e.g., Bodie, Merton, and Samuelson (1992). The total wealth is the sum of the financial wealth and the human wealth of the individual, i.e., the present value of future labor income. The desired risk exposure of the financial wealth is thus depending on the magnitude and risk attributes of human wealth and the demands for risky assets are adjusted accordingly to obtain the desired overall risk exposure. For a young individual, the human wealth is high and in order to diversify the total wealth, the individual will invest less in assets highly correlated with income and more in assets with low or negative correlation with income. Stocks and bonds are typically very little correlated with income, but house prices tend to be highly correlated with labor income, e.g., Cocco (2005) reports a correlation of 0.55. Young individuals should therefore have little or even negative exposure to house price risk, despite the fact that owner-occupied housing comes with a convenience yield due to the consumption services provided by the house and maybe even a positive expected capital gain. Here, a negative exposure may be implemented by taking a short position in house price linked financial assets. Young individuals are therefore inclined to rent rather than own the home they live in. Later in life, when the human wealth and thus the negative adjustment of housing investment are lower, the desired housing investment increases and will eventually reach and exceed the desired housing consumption, suggesting that the individual should buy his home and maybe even obtain a higher exposure to house prices by buying additional housing units and renting them out or by taking long positions in house price linked financial assets. In the final years, the desired housing investment again falls below the desired housing consumption, indicating a shift back to home rental. The optimal housing investment varies much more over the life-cycle than the optimal investments in bonds and stocks. Note that in order to find the described life-cycle pattern in house consumption and investment, we need a model like our model with both risky labor income, risky house price, and a positive correlation between the two.

Our comprehensive comparative statics show, among other things, that less risk-averse individuals should be more exposed to house price risk and engage in owner-occupied housing earlier in life. A higher house price volatility may lead to higher housing investment early in life, but will eventually lead to lower investments in housing. The level of income volatility and variations in income volatility hopefully capital gains so, by construction, the prices of REIT shares will be closely related to real estate prices. There are REITs specializing in different property types and in different locations. By the end of October 2009, 127 REITs were traded on NYSE with a total capitalization of $235 billion (see REITwatch November 2009 report on www.reit.com). Well-established REIT markets also exist in countries such as Japan, Canada, France, and the Netherlands, and are under development in many other countries, e.g., in Germany. See Cotter and Roll (2009) for an analysis of the risk and return characteristics of U.S. REITs. Tsai, Chen, and Sing (2007) report that REITs behave more and more like real estate and less and less like ordinary stocks. The CSI housing futures (and options on housing futures) are traded since 2006 at the Chicago Mercantile Exchange. The payoff of such a contract is determined by either a U.S. national home price index or by a home price index for one of 10 major U.S. cities; the indices were developed by Case and Shiller.
of life are very important for the magnitude and risk attributes of human wealth and, consequently, for the desired risk exposure of the financial investments. Our results illustrate the need for more information about how labor income risk typically varies with age. When we apply our setting with the life-cycle patterns in expected income growth estimated for different educational groups by Cocco, Gomes, and Maenhout (2005), we find that typical college graduates should invest less in the housing asset early in life and enter into owner-occupied housing later in life than typical less-educated individuals, but subsequently college graduates should be much more exposed to house prices either through financial investments in REITs and house price derivatives or by physically buying additional residential housing units and renting them out. The differences across educational groups are primarily caused by differences in the magnitude of their human wealth.

In order to derive closed-form solutions, we have to assume market completeness (cf., e.g., Liu 2007) so, in particular, the labor income stream has to be spanned by the traded assets. The correlations between an individual’s labor income and the returns on stocks and bonds are probably quite low. However, as mentioned above, labor income tends to be highly correlated with house prices so that the income spanning assumption is less unrealistic in our model with housing than in the models with labor income, but no housing, studied in the existing literature. Still it may not be possible to find a trading strategy in stocks, bonds, deposits, and houses that perfectly replicates the income risk. Without perfect spanning it seems impossible to derive the optimal investment strategy in closed-form. While the investment strategy we derive in this paper will then be sub-optimal, the results presented in Bick, Kraft, and Munk (2009) for a similar, though slightly simpler, model indicate that it will be near-optimal in the sense that the investor will suffer a small certainty-equivalent wealth loss by following the closed-form sub-optimal strategy instead of the unknown optimal strategy. The results we present are therefore relevant even without perfect spanning.

In our main model the individual can continuously and costlessly adjust both the housing consumption and the housing investment, but we also consider problems with limited flexibility in housing decisions. Changes in physical ownership of housing generate substantial transaction costs not included in our model. Of course, if the individual can trade in REITs or CSI housing contracts perfectly correlated with the relevant house prices, the housing investment position can be rebalanced frequently at low cost. Changing the rental position is also costly and unpractical. The case where both housing consumption and housing investment are continuously adjustable can therefore be seen as giving an upper bound on the life-time utility that the individual can realistically obtain. We investigate the importance of the frequency of adjustments of the housing consumption and housing investment in two ways. First, we derive an explicit solution to the problem where the individual consumes a constant level of housing services through life and adjusts the housing investment position continuously. We

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3The correlation between average labor income and the general stock market is usually estimated to be close to zero, see, e.g., Cocco, Gomes, and Maenhout 2005. More information about the relation between labor income and the stock market, see Davis and Willen (2000), Heaton and Lucas (2000), and Campbell and Viceira (2002).
find that the utility decrease due to the fixed housing consumption is equivalent to less than 0.25% of total wealth (corresponding to 1,400 USD) for our benchmark parameter values. While the individual would like to increase housing consumption over life as wealth increases and precautionary savings are reduced, she can compensate for a fixed housing consumption almost perfectly by adjusting the consumption of the perishable good. Second, we implement a Monte Carlo simulation procedure to compute the expected utility of an individual restricted to infrequent adjustments of (a) housing consumption, (b) housing investment, or (c) both. Again, we find that the wealth-equivalent loss is fairly small even when the housing positions are only adjusted every five years. Our results indicate that it is more important to adjust the housing investment position frequently than the housing consumption position, suggesting that a well-functioning market for REITs or other financial contracts related to house prices can lead to a non-negligible improvement in the welfare of individual investors.

Next we compare our setting and findings to the existing literature. The role of labor income in optimal consumption and portfolio choice has been studied in papers such as Bodie, Merton, and Samuelson (1992), Jagannathan and Kocherlakota (1996), Heaton and Lucas (1997), Chan and Viceira (2000), Munk (2000), Viceira (2001), Cocco, Gomes, and Maenhout (2005), Lynch and Tan (2009), Munk and Sørensen (2009), and Koijen, Nijman, and Werker (2010). They all ignore housing consumption and investment. Closed-form solutions are obtained only when the labor income is assumed to be spanned by the stocks and bonds included in the model, which is a highly unrealistic case. A few papers explore various implications of housing for consumption and portfolio choice, but ignore labor income. For example, Flavin and Yamashita (2002) incorporate some wealth consequences of housing positions into a mean-variance portfolio choice framework. Grossman and Laroque (1990) explore a model with a single illiquid durable consumption good (a house) which is traded with transaction costs. Damgaard, Fuglsbjerg, and Munk (2003) generalize the setting by having both perishable and durable (housing) consumption. They derive a closed-form solution without transaction costs and provide a mathematical and numerical analysis of the case with a proportional cost on house transactions. Their model does not allow for a distinction between housing consumption and investment.

Several recent papers include both labor income and housing in life-cycle decision problems, as we do in our paper. Campbell and Cocco (2003) study the mortgage choice in a life-cycle framework with stochastic house price, labor income, and interest rates. They do not allow housing investment to differ from housing consumption and, furthermore, fix the house size (the number of housing units), so they cannot address the interaction between housing decisions and portfolio decisions. Cocco (2005) considers a model in which house prices and aggregate income shocks are perfectly correlated. Also in his model housing consumption and housing investment cannot be disentangled, as renting is not possible and there are no house price linked financial assets traded. The individual can only enjoy the consumption benefits of a home by buying a house and is thus forced into home ownership. Since

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4Chan and Viceira (2000) and Viceira (2001) derive closed-form approximate solutions in a setting where income is not perfectly spanned.
there is a minimum choice of house size, a young individual has to tie up a large share of wealth in real estate and will invest little in stocks (also because of borrowing constraints and an imposed stock market entry cost). In our less restrictive model, the young individual can consume housing services via renting and will take a positive position in stocks. Cocco’s conclusion that house price risk crowds out stock holdings and can therefore help in explaining limited stock market participation does not carry over to our setting.

Yao and Zhang (2005) generalize Cocco’s setting to an imperfect correlation between income and house prices, and they show that there are substantial welfare gains from allowing renting and that the renting/owning decision changes the optimal investment strategy. In their model, the individual would prefer owning a house to renting, but cannot always do so because of constraints (e.g., a down payment is required to buy a house). If the individual decides to rent a house of a given size, that will be equal to his housing consumption and he will have zero wealth exposure to house price risk. If the individual decides to own a house, the size of the house determines his housing consumption and is identical to his housing investment position. Yao and Zhang (2005) find that home-owners invest less in stocks than home-renters. This is consistent with our findings that the optimal housing investments and stock investments are generally negatively related. Van Hemert (2009) generalizes the setting further by allowing for stochastic variations in interest rates and thereby introducing a role for bonds, and his focus is on the interest rate exposure and choice of mortgage over the life-cycle. We also allow stochastic interest rates.\(^5\) Compared to these models, we disconnect the housing consumption and investment positions further, as the individual can simultaneously rent and own, and his investment position can be higher or lower than the housing consumption by renting out part of the owned property or by investing in house price linked financial contracts. In the existing models, the housing investment position is closely linked to the demand for housing consumption, and that level of housing investment will affect the investments in the other risky assets to obtain the best overall level of risk-taking and exposure to different risks. In our setting, the housing investment is more freely determined and, hence, does not have similar repercussions for the stock and bond demand. As mentioned above, our simulation results indicate that access to well-functioning markets for financial assets linked to house price will lead to welfare gains that are non-negligible, although of a moderate magnitude. In related work, de Jong, Driessen, and Van Hemert (2008) conclude that the welfare gains from having access to housing futures are small, but their model ignores labor income risk, does not allow for renting, fixes the housing investment, and assumes utility only from terminal wealth. Other papers addressing various aspects of housing in individual decision making include Sinai and Souleles (2005), Li and Yao (2007), Cauley, Pavlov, and Schwartz (2007), Betermier (2009), and Corradin,\(^5\)

\(^5\) There is a large literature on the impact of stochastic interest rates on long-term portfolio choice in settings with no labor income or housing aspects. A partial list of references include Merton (1973), Sørensen (1999), Brennan and Xia (2000), Campbell and Viceira (2001), Munk and Sørensen (2004), Sangvinatsos and Wachtter (2005), and Detemple and Rindisbacher (2010).
Fillat, and Vergara-Alert (2010).  

Most of the papers listed above impose various realistic constraints on the investment decisions of the individual and/or allow labor income to have an unspanned risk component. Therefore, they solve the decision problems by numerical dynamic programming with a coarse discretization of time and the state space (Van Hemert (2009) is able to handle a finer discretization by relying on 60 parallel computers). This computational procedure is highly time-consuming and cumbersome, and little is known about the precision of the numerical results. The closed-form solutions derived in this paper are much easier to analyze, interpret, and implement and thus facilitate an understanding and a quantification of the economic forces at play.

To summarize our contribution, we derive a closed-form solution for the optimal life-cycle housing, consumption, and investment decisions in a rich model taking into account variability in labor income, interest rates, and the prices of houses, stocks, and bonds. We provide a thorough discussion of the structure of the optimal strategies and their sensitivity to key parameters. We show in a numerical example that our solution generates a life-cycle behavior with many realistic features. Our solution serves as a benchmark for similar decision problems, e.g., extensions to investment constraints, transaction costs, and/or unspanned income risk.

The remainder of the paper is organized as follows. Section 2 formulates and discusses the ingredients of our model and the utility maximization problem faced by the individual. Section 3 states, explains, and illustrates the optimal housing, consumption, and investment strategies in the case when housing decisions can be controlled continuously. Section 4 discusses the sensitivity of the optimal strategies to variations in key parameters. Section 5 investigates the effect of limiting the flexibility in revising housing decisions and provides estimates of the value of being able to make continuous revisions, e.g., via trade in financial contracts linked to house prices. Section 6 summarizes and concludes.

All proofs are collected in the appendices at the end of the paper.

2 The problem

The main elements of our modeling framework are specified as follows.

Consumption goods. The individual can consume two goods: perishable consumption and housing. The perishable consumption good is taken as the numeraire so that the prices of the housing good and of all financial assets are measured in units of the perishable consumption good.

\footnote{While we investigate individual decision making in the presence of housing wealth and human capital on individual decisions, the role of these two factors in equilibrium asset pricing have also been subject to recent theoretical and empirical research. Papers on the impact of housing decisions and prices on financial asset prices include Piazzesi, Schneider, and Tuzel (2007), Lustig and van Nieuwerburgh (2005), and Yogo (2006), whereas papers such as Constantinides, Donaldson, and Mehra (2002), Santos and Veronesi (2006), and Storesletten, Telmer, and Yaron (2004, 2007) focus on the interaction of labor income risk and asset prices.}
Financial assets. The individual can invest in three purely financial assets: a money market account (cash), a bond, and a stock (representing the stock market index). The return on the money market account equals the continuously compounded short-term real interest rate $r_t$, which is assumed to have Vasicek dynamics

$$dr_t = \kappa [\bar{r} - r_t] dt - \sigma_r dW_{rt},$$

(2.1)

where $W_r = (W_{rt})$ is a standard Brownian motion. The price of any bond (or any other interest rate derivative) is then of the form $B_t = B(r_t,t)$ with dynamics

$$dB_t = B_t [(r_t + \lambda_B \sigma_B(r_t,t)) dt + \sigma_B(r_t,t) dW_{rt}],$$

(2.2)

where $\sigma_B(r,t) = -\sigma_r B_r(r,t)/B(r,t)$ is the volatility and $\lambda_B$ the Sharpe ratio of the bond, which is identical to the market price of interest rate risk.\(^7\) In particular, if we introduce the notation

$$B_m(\tau) = \frac{1}{m} (1 - e^{-m\tau})$$

for any positive constant $m$, the time $t$ price of a real zero-coupon bond maturing at some date $T > t$ can be written as

$$B_t^T = e^{-a(T-t)-B_\kappa(T-t)r_t},$$

(2.3)

$$a(\tau) = \left[ \bar{r} + \frac{\lambda_B \sigma_r}{\kappa} - \frac{\sigma_r^2}{2\kappa} \right] (\tau - B_\kappa(\tau)) + \frac{\sigma_B^2}{4\kappa} B_\kappa(\tau)^2.$$  

(2.4)

An investor will not gain from trading in more than one bond in addition to the money market account.

The stock price $S_t$ has dynamics

$$dS_t = S_t \left[ (r_t + \lambda_S \sigma_S) dt + \sigma_S \left( \rho_{SB} dW_{rt} + \sqrt{1 - \rho_{SB}^2} dW_{St} \right) \right],$$

(2.5)

where $W_S = (W_{St})$ is a standard Brownian motion independent of $W_r$, $\sigma_S$ is the constant volatility and $\lambda_S$ the constant Sharpe ratio of the stock, and $\rho_{SB}$ is the constant correlation between the stock and the bond returns.

Houses. The individual can also buy or rent houses. A given house is assumed to be fully characterized by a number of housing units, where a “unit” is some one-dimensional representation of the size, quality, and location. Prices of all houses move in parallel. The purchase of $a$ units of housing costs $aH_t$; there are no transaction costs. The unit house price $H_t$ is assumed to have dynamics

$$dH_t = H_t \left[ (r_t + \lambda_H \sigma_H - r^{imp}) dt + \sigma_H \left( \rho_{HB} dW_{rt} + \rho_{HS} dW_{St} + \hat{\rho}_H dW_{Ht} \right) \right],$$

(2.6)

\(^7\)The Sharpe ratio of an asset is defined as the expected excess return (relative to the short rate) divided by the volatility of the asset. The term market price of risk refers to the compensation in terms of expected excess return per unit exposure to a given exogenous shock to the economy, which in our case means a given standard Brownian motion. For an asset that is only sensitive to one exogenous shock, the Sharpe ratio is identical to the market price of risk associated with that shock. For an asset sensitive to multiple shocks, the Sharpe ratio is a weighted sum of the market prices of risks associated with the shocks, where the weights are the correlations of the asset price with each shock.
where \( W_H = (W_{Ht}) \) is a standard Brownian motion independent of \( W_r \) and \( W_S \), \( \sigma_H \) is the constant price volatility and \( \lambda_H \) the constant Sharpe ratio of houses, \( \rho_{HB} \) is the constant correlation between house and bond prices, and

\[
\hat{\rho}_{HS} = \frac{\rho_{SH} - \rho_{SB}\rho_{HB}}{\sqrt{1 - \rho_{SB}^2}}, \quad \hat{\rho}_H = \sqrt{1 - \rho_{HB}^2 - \hat{\rho}_{HS}^2}
\]

where \( \rho_{SH} \) is the constant correlation between house and stock prices. Finally, \( r_{\text{imp}} \) is the imputed rent, i.e., the market value associated with the net benefits offered by a house (similar to the convenience yield of commodities), which is assumed to be constant as, e.g., in Van Hemert (2009).

The unit rental cost of houses is assumed to be proportional to the current unit house price, i.e., \( \nu H_t \) for some constant \( \nu \). This assumption serves to limit model complexity and is standard in the papers allowing for renting, cf. Yao and Zhang (2005) and Van Hemert (2009). For later use, define \( \lambda'_H = \lambda_H + (\nu - r_{\text{imp}})/\sigma_H \). By renting instead of owning the house, the individual can isolate the consumption role of housing.

The individual can obtain a pure investment exposure to house prices by buying and renting out housing units with a unit instantaneous return of

\[
dH_t + \nu H_t \, dt = H_t \left[ (r_t + \lambda'_H \sigma_H) \, dt + \sigma_H \left( \rho_{HB} dW_{rt} + \hat{\rho}_{HS} dW_{St} + \hat{\rho}_H dW_{Ht} \right) \right].
\]

(2.7)

If REITs or financial contracts linked to house prices are traded, they will also offer a pure investment exposure to house prices and must provide the return specified in (2.7).

We let \( \varphi_C t \) denote the number of housing units occupied by the individual at time \( t \), either through physical ownership or renting, and let \( \varphi_I t \) denote the number of housing units that the individual is financially exposed to, either through physical ownership or investments in house price linked financial assets.

Note that when an individual physically owns a house, a negative position in the money market account can be interpreted as an adjustable-rate mortgage, whereas a negative position in the long-term bond resembles a fixed-rate mortgage. In order to obtain closed-form solutions we do not limit borrowing to some fraction of the market value of the house owned.

**Labor income.** The individual receives a continuous and exogenously given stream of income from non-financial sources (e.g., labor) at a rate of \( Y_t \) with dynamics

\[
dY_t = Y_t \left[ \mu_Y (r_t, t) \, dt + \sigma_Y (t) \left( \rho_{YB} dW_{rt} + \hat{\rho}_{YS} dW_{St} + \hat{\rho}_Y dW_{Yt} \right) \right].
\]

(2.8)

For analytical tractability, we assume that there is no idiosyncratic shock to the income process so that the market is complete. The human capital or human wealth of the individual is the present value of the entire future labor income stream. In a complete market with risk-neutral probability measure \( Q \), the human capital is unique and given by the risk-neutral expectation of the discounted future income stream,

\[
L_t = E_t^{Q} \left[ \int_t^T e^{-\int_t^u r_s \, ds} Y_s \, ds \right].
\]

(2.9)
More specifically, we will assume that the individual retires from working life at time \( \tilde{T} \) and lives on until time \( T \geq \tilde{T} \), and

\[
\mu_Y(r,t) = \begin{cases} 
\bar{\mu}_Y(t) + br, & t \in (0, \tilde{T}), \\
0, & t \in [\tilde{T}, T], 
\end{cases} \\
\sigma_Y(t) = \begin{cases} 
\bar{\sigma}_Y, & t \in (0, \tilde{T}), \\
0, & t \in [\tilde{T}, T]. 
\end{cases}
\] (2.10)

Hence, in the retirement period \((\tilde{T}, T]\), the individual is assumed to receive a risk-free and constant income (in real terms), and we will assume that the income in retirement is equal to a fixed proportion (the “replacement ratio”) \( \Upsilon \) of the income at retirement, i.e.,

\[
Y_t = \Upsilon Y_{\tilde{T}}, \quad t \in (\tilde{T}, T], 
\] (2.11)

so that there is a jump in the income at retirement (downwards for the realistic case of \( \Upsilon < 1 \)). In the working phase, the expected percentage income growth is allowed to depend on time (age of the individual) and the interest rate level to reflect fluctuations of labor income over the life- and business cycle, cf., e.g., Cocco, Gomes, and Maenhout (2005) and Munk and Sørensen (2009). In the income dynamics (2.8), \( \rho_{YB} \) is the constant correlation between income growth and bond returns, and

\[
\hat{\rho}_{YS} = \frac{\rho_{SY} - \rho_{SB}\rho_{YB}}{\sqrt{1 - \rho_{SB}^2}}, \\
\hat{\rho}_{YH} = \sqrt{1 - \rho_{YB}^2 - \hat{\rho}_{YS}^2},
\]

where \( \rho_{SY} \) is the constant correlation between house and stock prices. Due to the completeness assumption the correlation between income growth and house prices follow from the other pairwise correlations,

\[
\rho_{YH} = \rho_{HB}\rho_{YB} + \hat{\rho}_{HS}\hat{\rho}_{YS} + \hat{\rho}_{H\hat{\rho}_{YH}}. 
\] (2.12)

Under the above assumptions, the risk-neutral income drift is \( \mu_Y(r_t, t) - \sigma_Y(t)\lambda_Y \), where \( \lambda_Y \) is the Sharpe ratio of a portfolio replicating the income process, which is a correlation-weighted combination of the risk premia \( \lambda_B, \lambda_S \), and \( \lambda'_H \), cf. (A.2) in Appendix A. The next theorem gives a closed-form solution for the human wealth \( L_t \).

**Theorem 2.1 (Human capital)** Assume that labor income is given by (2.8), (2.10), and (2.11).

(a) The human capital is

\[
L_t = \begin{cases} 
Y_t F(t, r_t), & t < \tilde{T}, \\
Y_{\tilde{T}} F(t, r_{\tilde{T}}), & t \in [\tilde{T}, T], 
\end{cases} 
\] (2.13)

where

\[
F(t, r) = \begin{cases} 
\int_t^{\tilde{T}} e^{-A(t,s)-(1-b)B_s(s-t)}r \, ds + \Upsilon \int_{\tilde{T}}^T e^{-\tilde{A}(t,s)-(B_s(s-t)-B_{s}(\tilde{T}-t))r} \, ds, & t < \tilde{T}, \\
\Upsilon \int_t^{\tilde{T}} e^{-a(s-t)-B_s(s-t)}r \, ds, & t \in [\tilde{T}, T]. 
\end{cases}
\] (2.14)

Here \( A(t, s) \) and \( \tilde{A}(t, S) \) are deterministic functions stated in Appendix A.
(b) The expected future income rate is given by
\[
E_0[Y_t] = Y_0 \exp \left\{ \int_0^t \tilde{u}_Y(u) \, du + b \mu_0 B_t(t) + b \left( \frac{b \sigma_r^2}{2 \kappa^2} \right) (t - B_t(t)) - \frac{b^2 \sigma_r^2}{4 \kappa} B_t(t)^2 - b \sigma_r \rho_{YB} \int_0^t \sigma_Y(u) (t - u) \, du \right\}, \quad t \leq \tilde{T},
\]
and \( E_0[Y_t] = \Upsilon E_0[Y_{\tilde{T}}] \), \( t \in [\tilde{T}, T] \). The expected future human capital is
\[
E_0[L_t] = \begin{cases} 
E_0[Y_t] \left( \int_t^\tilde{T} e^{-A(t,s) - f_1(t,s,s)} \, ds + \Upsilon \int_{\tilde{T}}^T e^{-A(t,s) - f_1(t,s,\tilde{T})} \, ds \right) \approx E_0[Y_t]F(t, \tilde{r}), & t < \tilde{T}, \\
E_0[Y_{\tilde{T}}] \int_t^\tilde{T} e^{-a(s-t) - B_c(s-t) f_2(t,s)} \, ds \approx E_0[Y_{\tilde{T}}] \int_t^T e^{-a(s-t) - B_c(s-t) \tilde{r}} \, ds, & t \in [\tilde{T}, T],
\end{cases}
\]
where \( f_1 \) and \( f_2 \) are deterministic functions stated in Appendix A.

For a proof, we refer the reader to Appendix A.

Wealth dynamics. The individual’s tangible wealth at any time \( t \) is denoted by \( X_t \) and defined as the value of his current position in the money market account, the bond, the stock, and REITs, plus the value of the house owned by the individual. Let \( \pi_{St} \) and \( \pi_{Bt} \) denote the fraction of tangible wealth invested in the stock and the bond, respectively, at time \( t \). The wealth invested in the money market account is then \( X_t(1 - \pi_{St} - \pi_{Bt}) - (\varphi_{at} + \varphi_{Rt})H_t \). Finally, let \( c_t \) denote the rate at which the perishable good is consumed at time \( t \). The dynamics of tangible wealth is then
\[
dX_t = \pi_{St} X_t \frac{dS_t}{S_t} + \pi_{Bt} X_t \frac{dB_t}{B_t} + [X_t(1 - \pi_{St} - \pi_{Bt}) - (\varphi_{at} + \varphi_{Rt})H_t] r_t \, dt \\
+ \varphi_{at} \lambda_t H_t + \varphi_{Rt} (dH_t + \nu H_t \, dt) - c_t \, dt - \gamma X_t \, dt \\
= [X_t(r_t + \pi_{St} \lambda_t s + \pi_{Bt} \lambda_t B_t) + \varphi_{at} \lambda_t H_t H_t - \varphi_{at} \nu H_t - c_t + Y_t] \, dt \\
+ (\pi_{St} X_t \rho_{SB} s + \pi_{Bt} X_t \rho_{SB} B_t + \varphi_{at} \lambda_t \rho_{HB} H_t) \, dW_t \\
+ \left( \pi_{St} X_t \sigma_s \sqrt{1 - \rho_{SB}^2 + \varphi_{at} \lambda_t \rho_{HS} H_t + \varphi_{at} \lambda_t \rho_{HS} H_t} \right) \, dW_t + \varphi_{at} \lambda_t H_t \, dW_{Ht} + \varphi_{at} \lambda_t \rho_{HS} H_t \, dW_{Ht},
\]
where
\[
\varphi_{Ct} \equiv \varphi_{at} + \varphi_{rt}, \quad \varphi_{Ht} \equiv \varphi_{at} + \varphi_{Rt}, \quad (2.18)
\]
so that \( \varphi_{Ct} \) is the total units of housing occupied by (and thus providing housing services to) the individual and \( \varphi_{Ht} \) is the total units of housing invested in either physically or indirectly through REITs. The wealth dynamics and the welfare of the individual are thus only affected by \( \varphi_{Ct} \) and \( \varphi_{Ht} \) so that, in general, there will be one degree of freedom. To obtain a unique solution we will have to fix one of the three control variables \( \varphi_{at}, \varphi_{rt}, \) and \( \varphi_{lt} \).

Preferences. We assume a time-additive Cobb-Douglas style utility so that the value function associated with the life-cycle consumption and investment problem is
\[
J(t, x, r, h, y) = \left( e^{-\delta(x-t)} \right) \sup_{(c, \pi, \varphi, \pi_{SB}, \pi_{BS}) \in A_t} \mathbb{E}_t \left[ \int_t^T e^{-\delta(u-t)} \frac{1}{1 - \gamma} \left( c^{\beta} \rho_{C}^{\gamma} \right)^{1-\gamma} ds + e^{-\delta(T-t)} \frac{1}{1 - \gamma} X_T^{1-\gamma} \right],
\]
(2.19)
where \( \delta \) is the subjective time preference rate, \( \gamma > 1 \) is the relative risk aversion, \( \beta \in (0, 1) \) is the relative weighting of the two consumption goods, and \( \varepsilon \geq 0 \) is the preference weight on terminal wealth. In (2.19), it is understood that the expectation is conditioned on \( X_t = x, r_t = r, H_t = h, \) and \( Y_t = y, \) and \( \mathcal{A}_t \) is the set of all admissible control processes over the time interval \([t, T]\). Constraints on the controls are thus reflected by \( \mathcal{A}_t \). Similar preferences are assumed in other recent papers modeling related two-good utility maximization problems, cf. Cocco (2005), Yao and Zhang (2005), and Van Hemert (2009). Note that a unit of housing consumed is assumed to contribute identically to the direct utility whether owned or rented.

### 3 The solution

Assume for now that the individual can continuously and costlessly adjust both the number of housing units consumed and the number of housing units invested in. We shall refer to this situation as “fully flexible housing decisions.” Due to (2.18), we can assume that the individual never has any direct ownership of housing units but continuously adjusts the investment in REITs to obtain the desired housing investment level and continuously adjust the number of housing units rented to achieve the desired housing consumption level. Alternatively, we can disregard REITs and assume a continuously adjusted direct ownership of housing units (admittedly, that may involve substantial transactions costs excluded from the theoretical framework of this paper), as well as a continuously adjusted renting position. We will first state and interpret the solution, then illustrate key properties of the solution for a set of benchmark parameters, and finally discuss selected comparative statics.

#### 3.1 Statement and interpretation of the solution

The model is fairly complex with four state variables (wealth, interest rate, house price, and income), but due to the assumed market completeness, the affine dynamics, and the assumption of no constraints on the controls, we are able to solve the problem in closed form as summarized in the following theorem. The proof is given in Appendix B.

**Theorem 3.1** With fully flexible housing decisions, the value function is given by

\[
J(t, x, r, h, y) = \frac{1}{1 - \gamma} g(t, r, h)^\gamma (x + yF(t, r))^{1-\gamma},
\]

where \( F \) is defined in (2.14) and

\[
g(t, r, h) = \varepsilon \frac{1}{1 - \gamma} e^{D_1(T-t) - \frac{\beta + 1}{\beta} B_s(T-t)r} + \frac{\eta \mu}{1 - \beta} h^k \int_t^T e^{-d_1(u-t) - \beta \frac{\beta + 1}{\beta} B_s(u-t)r} \, du.
\]
Here $k = (1 - \beta)(1 - 1/\gamma)$, $\eta = \beta^{1/\gamma} \left( \frac{\beta \nu}{1 - \beta} \right)^{-1}$,

$$
D_\gamma(\tau) = \left( \frac{\delta}{\gamma} + \frac{\gamma - 1}{2\gamma^2} \lambda^\top \tilde{X} \right) \tau + \left( \bar{r} + \frac{\gamma - 1}{\gamma^2} \sigma_r \lambda_B \right) \frac{\gamma - 1}{\gamma} \left( \tau - B_\kappa(\tau) \right)
$$

$$
- \frac{1}{\gamma} \left( \frac{\gamma - 1}{\gamma^2} \right)^2 \left( \tau - B_\kappa(\tau) - \frac{\kappa B_\kappa(\tau)}{\gamma} \right),
$$

and

$$
d_1(\tau) = \left( \frac{\delta}{\gamma} + \frac{\gamma - 1}{2\gamma^2} \lambda^\top \tilde{X} - k \left( \frac{1}{\gamma} \lambda_H \lambda_H - \nu + \frac{1}{2}(k - 1)\sigma_H^2 \right) \right) \tau
$$

$$
+ \beta \left( \bar{r} + \frac{\gamma - 1}{\gamma^2} \sigma_r \lambda_B \right) - \frac{k \sigma_r \gamma H \lambda H}{\gamma} \left( \tau - B_\kappa(\tau) \right)
$$

$$
- \frac{1}{\gamma} \left( \frac{\gamma - 1}{\gamma^2} \right)^2 \left( \tau - B_\kappa(\tau) - \frac{\kappa B_\kappa(\tau)}{\gamma} \right),
$$

where $\lambda^\top \tilde{X} = \lambda_B \xi_B + \lambda_S \xi_S + \lambda_H \xi_I$. The optimal controls are given by

$$
\pi_S = \frac{1}{\gamma} \frac{\xi_S + yF}{\sigma_S},
$$

$$
\pi_B = \frac{1}{\gamma} \frac{\xi_B + yF}{\sigma_B} - \frac{\sigma_Y(t) \xi_B yF}{\sigma_B} - \frac{\sigma_Y(t) \xi_B yF}{\sigma_B} - \frac{\sigma_r yF \xi_B}{\sigma_B} - \frac{\sigma_r yF \xi_B}{\sigma_B},
$$

$$
\varphi_I = \frac{1}{\gamma} \frac{\xi_I + yF}{\sigma_I},
$$

$$
c = \frac{\gamma \sigma_B y}{\gamma - \beta} \frac{k \tau x + yF}{g},
$$

$$
\varphi_C = \frac{\gamma \sigma_B y}{\gamma - \beta} \frac{k \tau x + yF}{g}.
$$

In all these expressions, $y$ is to be replaced by $Y_\tau$ when $t \in [\hat{T}, T]$.

The constants $\xi_B, \xi_S, \xi_I$ are defined in (B.14)-(B.16) in Appendix B in terms of the market prices of risk $\lambda_B, \lambda_S, \lambda_H$ and the pairwise correlations between prices on the bond, the stock, and the house. The constants $\xi_B, \xi_S, \xi_I$ are defined in (A.3)-(A.5) in Appendix A in terms of the pairwise correlations between the bond, the stock, the house, and the labor income.

This form of the value function has also been found in many simpler cases. The total initial wealth of the individual is the sum of the tangible wealth $x$ and the human capital which, according to Theorem 2.1, equals $yF(t, r)$ with $F$ given by (2.14). As in the existing solutions to similar, but simpler, problems studied in the literature, the $g$ function is determined by the assumed asset price dynamics and will generally depend on variables sufficient to describe relevant variations in the investment opportunity set; see, e.g., Liu (2007). Long-term investors will generally want to hedge variations in investment opportunities as captured by the short-term interest rate and the maximum Sharpe ratio, which together define the location of the instantaneous mean-variance efficient frontier, cf. Nielsen and Vassalou (2006). Since $\lambda_B, \lambda_S,$ and $\lambda_H$ are assumed constant, there are no variations in the maximum Sharpe ratio, so the short-term interest rate alone drives investment opportunities. In addition, a long-term investor who can control her consumption of multiple goods affecting her utility will want to hedge variations in the relative prices of those consumption goods. In our model, the relative price of the two consumption goods is given by $H_t$. This explains why $g$ is a function of $r$ and $h$ in our setting.
The first terms in (3.5), (3.6), and (3.7) reflect the speculative demand well-known from the static mean-variance analysis and are determined by wealth, relative risk aversion, volatilities, correlations, and the market prices of risk.

The second terms in the equations reflect an adjustment of the investments to the risk profile of human wealth. We can think of the individual first determining the desired exposure to all the exogenous shocks—i.e., the standard Brownian motions $W_r$, $W_s$, and $W_H$—and then adjusting for the exposure implicit in the human wealth in order to obtain the desired exposure of the explicit investments towards the shocks. The appropriate adjustment is determined by the instantaneous correlations between the assets and the labor income through the constants $\zeta_B, \zeta_S, \zeta_I$. In addition, human wealth is discounted future labor income and therefore interest rate dependent. From (2.14), it follows that before retirement

$$F_r(t, r) = -(1 - b) \int_t^\bar{T} B_n(s - t)e^{-A(t,s) - (1-b)B_n(s-t)r} ds$$

$$- \Upsilon \int_t^\bar{T} \left(B_n(s - t) - bB_n(\bar{T} - t)\right)e^{-\tilde{A}(t,s) - (B_n(s-t) - bB_n(\bar{T} - t))r} ds,$$

whereas in retirement

$$F_r(t, r) = -\Upsilon \int_t^\bar{T} B_n(s - t)e^{-a(s-t) - B_n(s-t)r} ds.$$

Hence, as long as the interest rate sensitivity of the expected income growth rate $b$ is below 1, human wealth is decreasing in the interest rate level and is thus similar to an investment in the bond. If the expected income growth rate is strongly pro-cyclical, i.e., $b$ is sufficiently greater than 1, the human wealth before retirement will be increasing in the interest rate corresponding to an implicit short position in the bond, which is corrected for by a positive explicit demand for the bond. For further discussion of this point, see Munk and Sørensen (2009). The time-dependence of human wealth, as reflected by the function $F(t, r)$, induces a non-constant optimal stock portfolio weight. To be consistent with the popular advice of having “more stocks when you have a long investment horizon”, we need $\xi_S > \gamma \sigma_Y(t) \zeta_S$, which obviously depends on the level of risk aversion and the income volatility, but also on the market prices of risk and numerous correlations embedded in $\xi_S$ and $\zeta_S$.

The last term in (3.6) hedges against variations in future investment opportunities which are summarized by the short-term interest rate and thus hedgeable through a bond investment. In all our computations, we find $g_r/g < 0$ so that the intertemporal hedge demand for the bond is positive consistent with intuition and the existing literature. Finally, the last term in (3.7) represents a hedge against variations in the house price. When house prices increase, the costs of future housing increase. To compensate for that, the individual can invest more in houses so that an increase in house prices will also increase her wealth. Consistent with that interpretation, we find $g_h/g$ to be positive in all our computations below. An investment in a house is a hedge against future costs of housing consumption, as emphasized by Sinai and Souleles (2005).
Concerning consumption, the optimal total expenditure on the two consumption goods is

$$c + \nu h \phi C = \delta^{1/\gamma} \beta^{\theta(1/\gamma - 1)} \left( \frac{\nu}{1 - \beta} \right)^k h^k (x + yF)g^{-1}.$$  

The individual distributes the total consumption expenditure to perishable consumption and housing consumption according to the relative weights $\beta$ and $1 - \beta$ of the goods in the preference specification. The optimal spending on each good is a time- and state-dependent fraction of the total wealth $x + yF$.

It can be shown that (substitute the above expression for total consumption into (B.17)), using the optimal strategies, the dynamics of total wealth $W_t = X_t + Y_t F(t, r_t)$ will be

$$\frac{dW_t}{W_t} = \left( r_t + \frac{1}{\gamma} \tilde{\lambda}^T \tilde{\lambda} - \sigma_r \lambda_B \frac{g_r}{g} + \sigma_H \lambda_H \frac{g_h}{g} - \frac{\eta \nu}{1 - \beta} H_t^k g^{-1} \right) dt + \frac{1}{\gamma} \tilde{\lambda}^T dW_t - \frac{g_r}{g} \sigma_r dW_{rt} + H_t \frac{g_h}{g} \sigma_H H_t dW_t,$$

where

$$\tilde{\lambda}^T = \left( \lambda_B, \frac{\lambda_S - \rho_{SB} \lambda_B}{\sqrt{1 - \rho_{SB}^2}}, \frac{1}{\rho_H} \left[ \lambda_H' - \rho_{SH} - \rho_{SB} \rho_{HB} \frac{\lambda_S}{1 - \rho_{SB}^2} - \rho_{HB} - \rho_{SB} \rho_{HS} \frac{\lambda_B}{1 - \rho_{SB}^2} \right] \right)$$

is the market price of risk vector associated with the standard Brownian motion $W = (W_r, W_S, W_H)^T$, and $\tilde{\rho}_H = (\rho_{HB}, \rho_{HS}, \rho_{HH})^T$. The term $\frac{1}{\gamma} \tilde{\lambda}^T dW_t$ reflects the optimal risk taking in a setting with constant investment opportunities and the term $\frac{1}{\gamma} \tilde{\lambda}^T \tilde{\lambda}$ in the drift gives the compensation in terms of excess expected returns for that risk. The shock terms $- \frac{g_r}{g} \sigma_r dW_{rt}$ and $H_t \frac{g_h}{g} \sigma_H H_t dW_t$ are the optimal adjustments of the exposure to interest rate risk and house price risk, respectively, due to intertemporal hedging of shifts in investment opportunities, again with appropriate compensation in the drift of wealth. The ratios $g_r/g$ and $g_h/g$ involve the risk aversion of the individual.

### 3.2 Benchmark parameter values

In our quantitative illustrations of the solutions, we will use the benchmark parameter values listed in Table 1. For each group of parameters, the exogenously specified parameters are listed above the dashed line, whereas the parameters below the dashed line follow from the others. Our benchmark parameter values are roughly in line with those used in similar studies referred to in the introduction.

Whenever we need to use levels of current or future house prices, wealth, labor income etc., we use a unit of USD 1 scaled by one plus the inflation rate in the perishable consumption good. In addition to the benchmark parameters, we assume an initial tangible wealth of $X_0 = 20,000$ and an initial income rate of $Y_0 = 20,000$ unless otherwise mentioned. The current value of the short-term interest rate is set identical to the long-term average, $r = \bar{r}$.

The parameters governing the dynamics of the short-term real risk-free interest rate reflect a relatively slowly moving rate around a long-term level of 2%. Whenever we need to specify the bond in which the individual invests, we take it to be a (real) zero-coupon bond maturing 20 years later. The volatility of the bond is 7.4% and the market price of interest rate risk implies that the expected return
of the bond is 2.7%, when the short rate is at its long-term level. For the stock, we apply a volatility of 20% and an expected return of 5% in excess of the short rate, assuming a standard reduction of the historical equity premium as discussed by, e.g., Fama and French (2002).

For concreteness we think of houses as being fully represented by the number of square feet (of “average quality and location”) and will use an initial value of \( h = 250 \) corresponding to USD 250,000 for a house of 1,000 square feet. The imputed rent and the rental rate are both set at 5%. In particular, the initial monthly rent for 1,000 square feet is a little more than USD 1,000. The volatility of house prices is fixed at 12%, which is in line with the approximately 14% estimated by Flavin and Yamashita (2002) and the 10% used by Yao and Zhang (2005), and above the city index volatilities of 3-6% estimated by Goetzmann and Spiegel (2002). The market price of house price risk is set so that the expected growth rate of house prices is 1.0% per year, when the short rate is at its long-term average, exactly the values assumed by Cocco (2005) and in the range estimated by Goetzmann and Spiegel (2002).

In our main illustrations we assume that the interest rate independent part of the expected income growth rate \( \bar{\mu}_Y \) is constant until retirement. This allows us to focus on understanding the impact of the state variables and their interactions on the life-cycle behavior while disregarding the more mechanical time-dependence. In Section 4 we will explore the consequences of the age-dependence in income growth estimated by Cocco, Gomes, and Maenhout (2005). We assume that the income growth rate is increasing in the risk-free rate with a slope of 0.5, which is in line with the estimates reported by Munk and Sørensen (2009). When the risk-free rate is at its long-term level, the expected real income growth rate is then 2% per year. The income volatility is assumed to be 7.5% in accordance with the 5.8%, 7.4%, and 10.6% estimated for three educational groups by Cocco, Gomes, and Maenhout (2005). As Yao and Zhang (2005), we assume an income replacement ratio of 60%, which is slightly below the replacement ratios of 68-93% estimated by Cocco, Gomes, and Maenhout (2005). One reason for the lower replacement ratio is that retirees typically have to spend a large share of their income on medical expenses, leaving less income for standard consumption goods and housing. Furthermore, we do not explicitly model the mortality risk which is increasing in age and thus lowers the expected income in retirement.

We consider a relatively young individual with 30 years to retirement followed by 20 years in retirement. The relative risk aversion is 4 and the time preference rate is 3%, both standard values in the literature. We follow Yao and Zhang (2005) and assume that perishable and housing consumption enter the utility function with an 80/20 weighting, which is consistent with observed household housing expenditures, cf. a report by U.S. Department of Labor (2003). The parameter \( \tilde{\gamma} \) listed in the table is used only in Section 5.

Concerning the correlation structure, we assume that the income rate is instantaneously uncorrelated with both the stock index and the bond, in accordance with estimates of Cocco, Gomes, and Maenhout (2005), Munk and Sørensen (2009), and others. Also the stock index and the bond are as-
sumed to have a zero instantaneous correlation. We assume that the house price has a relatively high positive correlation with both the stock, the bond, and the labor income rate. The correlations have to be set so that the market is complete. The values are chosen so that our house-income correlation (0.57) is very close to the empirical estimate (0.55) of Cocco (2005) (who assumes perfect correlation in his illustrations). The positive correlation between house prices and bond prices is natural since decreasing interest rates will imply both higher bond prices and cheaper financing of houses, which should quickly capitalize in higher house prices. A positive correlation between house prices and stocks is natural since rising stock prices may increase the housing demand of stock-holders. Conversely, an increase in house prices leads to a higher total wealth of home owners which may induce them to increasing their stock positions. If we think of the housing investment as a direct physical ownership of housing units, then the high positive correlation between the stock and the house price can be obtained by tilting the stock index towards REITs. The estimation of Van Hemert (2009) also results in a significant positive correlation between the house price and the stock index and between the house price and bond prices.

3.3 Illustration of the solution for benchmark parameters

First, we consider consumption. The optimal spending on housing consumption equals the perishable consumption multiplied by the factor $(1 - \beta)/\beta$, which is 0.25 with our benchmark parameters. With no utility from terminal wealth, i.e., $\varepsilon = 0$, we have $g(t, r, h) = \frac{\eta \nu}{1 - \beta} W(t, r)$, where $G(t, r) = \int_t^\tau e^{-d_1(u-t) - \beta(1 - \frac{1}{\gamma})B_s(u-t)r} du$. The optimal spending on consumption goods will be

$$c_t = \beta \frac{W_t}{G(t, r_t)}, \quad \varphi_C t = (1 - \beta) \frac{W_t}{G(t, r_t)},$$

and, in particular, independent of the current house price. The first-order derivatives of $g$ that enter the optimal portfolio weights are then

$$\frac{g_h}{g} = \frac{k}{h} = h^{-1}(1 - \beta) \frac{\gamma - 1}{\gamma}, \quad \frac{g_r}{g} = -\beta \frac{\gamma - 1}{\gamma} D(t, r),$$

where $D(t, r) = \left(\int_t^\tau B_s(u-t) \exp\{-d_1(u-t) - \beta \frac{\gamma - 1}{\gamma} B_s(u-t)r\} du\right) / G(t, r)$. The dynamics of total wealth in (3.10) simplifies to

$$\frac{dW_t}{W_t} = \left(r_t + \frac{1}{\gamma} \bar{\lambda}^* \bar{\lambda} + \sigma_r \lambda_B \beta \frac{\gamma - 1}{\gamma} D(t, r_t) + k\sigma_H \lambda_H^r - G(t, r_t)^{-1}\right) dt$$

$$+ \frac{1}{\gamma} \bar{\lambda}^* dW_t + \beta \frac{\gamma - 1}{\gamma} \sigma_r D(t, r_t) dW_t + k\sigma_H \sigma_H^r dW_t.$$ 

In Appendix B we compute the time 0 expectation of $W_t/G(t, r_t)$, which leads to the expected spending on the two goods over the life-cycle. Appendix B also contains similar results for $\varepsilon > 0$.

Figure 1 illustrates the expected consumption pattern over the life-cycle. The figure shows the expected expenditure on each of the two consumption goods on the left scale. The expected perishable
consumption grows from around 21,000 to 57,000 USD per year over the assumed 50 year horizon. The expected expenditure on housing consumption is again just a \((1 - \beta)/\beta\) multiple, i.e., 25\%, of the expected perishable consumption. The expectation of the house price on the left-hand side of the second expression in (3.11) is
\[E_0[H_t] = H_0 \exp \left\{ (r_0 B_b(t) + (\lambda_H \sigma_H - r_{\text{imp}}) t + \left( r + \frac{\sigma_r^2}{2 \kappa^2} - \frac{\sigma_r \sigma_H \rho_{HB}}{\kappa} \right) (t - B_b(t)) - \frac{\sigma_r^2 B_b(t)^2}{4 \kappa} \right\}. \]
(3.14)

Now we can estimate the expected number of housing units consumed as \((1 - \beta)\frac{E_0[W_t]}{\nu E_0[H_t]}\) (the high precision of this approximation has been verified by Monte Carlo simulations). This is illustrated by the dashed curve using the right-side scale in Figure 1. The expected number of housing units consumed grows from about 420 to 800 over the 50 year remaining life-time. Recall that a housing unit can be thought to represent one square foot of housing (of “average quality”) so the above numbers (square feet per person) are of a reasonable magnitude.

Next, we consider the optimal investments. Following (3.5)–(3.7), the optimal investments as fractions of total wealth are given by
\[\hat{\pi}_S \equiv \frac{\pi_S}{x+yF} = \frac{1}{\gamma} \frac{\xi_S}{\sigma_S} \frac{\sigma_Y(t) \xi_S}{yF} \frac{yF}{x+yF}, \]
(3.15)
\[\hat{\pi}_B \equiv \frac{\pi_B}{x+yF} = \frac{1}{\gamma} \frac{\xi_B}{\sigma_B} \frac{\sigma_Y(t) \xi_B}{\sigma_B} \frac{yF}{x+yF} - \frac{\sigma_r}{\sigma_B} \frac{F_r}{F} - \frac{\sigma_r g_r}{\sigma_B g}, \]
(3.16)
\[\hat{\pi}_I \equiv \frac{\pi_I}{x+yF} = \frac{1}{\gamma} \frac{\xi_I}{\sigma_H} \frac{\sigma_Y(t) \xi_I}{\sigma_H} \frac{yF}{x+yF} + \frac{h g_h}{g}. \]
(3.17)

Figure 2 shows how these fractions vary with the human wealth to total wealth ratio, \(yF/(x+yF)\). It is clear that the optimal fraction of wealth invested in the bond also depends on the level of the interest rate through the terms \(F_r/F\) and \(g_r/g\), but these terms vary relatively little over the life-cycle. Similarly, the optimal fraction of wealth invested in the house includes the hedge term \(h g_h/g\), which is independent of the house price for the benchmark case of no bequest utility, cf. (3.12). Moreover, it varies very little with the level of the interest rate. Consequently, the ratio of human wealth to total wealth is the main driver of the life-cycle behavior of the optimal fractions of total wealth invested in the different assets. Another important effect is caused by the assumed jump in the income volatility \(\sigma_Y(t)\) at retirement from the 7.5\% assumed as the benchmark to zero. This eliminates the terms adjusting for income risk in the above expressions for the optimal investments. In Figure 2 the solid lines apply before retirement and the dashed lines in retirement. With our benchmark values of parameters and state variables, the human-to-total wealth ratio is initially about 95\% and decreases, in expectation, roughly linearly over life to about 53\% at retirement and to about 31\% a year before the terminal date.

\[\text{[Figure 2 about here.]}\]

The house price dynamics (2.6) implies that \(H_t = H_0 \exp \left\{ \int_0^t r_u du + (\lambda_H \sigma_H - r_{\text{imp}} - \frac{1}{2} \sigma_r^2) t + \int_0^t \sigma_H \rho_{HB} dW_u \right\}. \]
Substituting (A.9) and taking expectations, we find the expected house price stated in the text.
The fraction of total wealth invested in the *stock* consists of a constant speculative position of 3.9% and an adjustment for labor income which for a constant income volatility of 7.5% increases linearly from 0 to 32.7% with the relative importance of human wealth increases from 0% to 100%. The income-motivated adjustment of the stock position is positive since the auxiliary parameter $\zeta_S$ is negative. At retirement the income adjustment disappears leading to a sudden decline in the stock investment. In retirement, the stock demand stays constant at the 3.9% of total wealth.

The fraction of total wealth invested in the *bond* consists of four terms: (i) a constant speculative position of -62.6%, (ii) an adjustment due to the instantaneous correlation of income with financial assets, which for a fixed income volatility of 7.5% increases linearly from 0 to 115.7% with the human wealth to total wealth ratio, (iii) an adjustment due to the dependence of human wealth on the interest rate varying from 0 to -42.4% as the human/total wealth ratio goes from 0 to 1, and (iv) an intertemporal hedge against interest rate risk which amounts to 49.2% no matter how the total wealth is decomposed. The total bond demand varies from -13.4% to 59.8% as the human/total wealth is varied from 0 to 1. At retirement, when future income turns risk-free, the term (ii) vanishes and the total bond demand drops quite significantly.

The fraction of total wealth invested in houses (physically or financially) consists of a constant speculative demand of 91.2%, an income-adjustment term which for a fixed income volatility of 7.5% varies from 0 to -109.2% as the human/total wealth ratio goes from 0 to 1, and an intertemporal hedge against house price risk equal to 15% independent of wealth composition. The hedging motive for house investments emphasized by Sinai and Souleles (2005) is thus clearly dominated in magnitude by the speculative demand and the income-adjustment term. The large negative income adjustment is due to the large positive correlation between labor income and house prices. The initially large human wealth has a risk structure similar to that of a housing investment and therefore replaces a direct investment in housing. As human wealth declines, the direct investment in housing is increased to maintain the desired risk-return tradeoff. The total investment in houses varies from 106.2% with no human wealth to -3.0% with only human wealth. At retirement, the large negative income-adjustment term disappears leading to a very dramatic increase in the investment demand for housing.

Figure 3 shows how the expected total wealth, human wealth, and financial wealth vary over the life-cycle again assuming an initial financial wealth of $X_0 = 20,000$ and an initial labor income rate of $Y_0 = 20,000$. The graph is produced using an approximation of the expected total wealth $E_0[W_t]$ stated

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9Here, the component (iii) depends on $F_r/F$, the relative sensitivity of human wealth with respect to the interest rate. The numbers just reported and used to generate the figure assume 30 years to retirement in which case $F_r/F \approx -2.1$, but the ratio goes to 0 as retirement is approaching which will slightly increase the fraction of total wealth invested in the bond. The component (iv) depends on the ratio $g_r/g$, which is approximately $-2.42$ for a remaining life-time of 50 years. The ratio approaches zero relatively slowly as time passes, which leads to a lower hedge-motivated bond position.

10If we had chosen a different bond (or another interest rate dependent asset, e.g., a bond future), the optimal investment in that asset would have been a multiple of the optimal investment in the 20-year bond in order to obtain the same overall exposure to the shocks to the short-term interest rate.
in (B.20) and the approximation \( F(t, \bar{r})E_0[Y_t] \) of expected human capital, where the expected income is given by (2.15) in Theorem 2.1. Monte Carlo experiments have shown that these approximations are very precise. The expected financial wealth is computed residually. Human wealth dominates financial wealth throughout the working life and even a few years into retirement. The assumed income received in retirement imply relatively low “voluntary” savings on the individual’s own accounts. In the last 15 years, financial wealth is decreasing as some of the savings are spent on consumption goods. Both total wealth and financial wealth display the hump-shaped pattern often found in life-cycle studies.

[Figure 3 about here.]

The expected amounts invested in the different assets over the life-cycle now follows by combining the portfolio weights illustrated in Figure 2 and the wealth patterns in Figure (3). The results can be seen in Figure 4. The amount invested in the stock is roughly constant at 200,000 USD in the first 20 years, but then decreases somewhat until retirement because of the decline in the human wealth. At retirement, the expected stock investment drops from about 142,000 to about 26,000 after which it slowly falls towards zero at the terminal date. Note that the life-cycle profile is consistent with the standard asset allocation advice of having more stocks when young.

The individual is initially heavily invested in bonds, since the positive interest rate hedge term and the income-adjustment term more than outweigh the negative speculative demand. With the benchmark parameters, the bond demand is more sensitive to human wealth than the stock demand, so the expected bond investment goes down from the initial 323,000 USD to 19,000 USD immediately before retirement. At retirement, the expected bond investment drops to -394,000 USD after which it gradually increases towards zero. The life-cycle profile of the bond investment is very different from that found by Munk and Sørensen (2009) in a setting ignoring housing investment and consumption, mainly because of the significant negative speculative demand for bonds in our setting which is a product of the correlation structure and the relatively low Sharpe ratio on the bond compared to the stock and the house investment.

The housing investment starts out at about 5,000 USD and increases steadily to 320,000 USD just before retirement in line with the decline in human wealth. At retirement, all income risk is resolved and human wealth suddenly resembles the bond much more than the housing asset. Therefore, the housing investment takes a significant upwards jump to around 706,000 USD and then declines to zero as the terminal date is approaching. The high housing investment in retirement is due both to the attractive risk-return tradeoff of housing and the hedging against unfavorable shifts in future housing expenditures. The negative position in the long-term bond in retirement can be interpreted as a fixed-rate mortgage financing of the housing investment. Before retirement, the optimal strategy involves borrowing via the money market account, and a part of that borrowing can be interpreted as an adjustable-rate rate mortgage financing of the housing investment. These findings are consistent with the results of Van Hemert (2009).
Figure 5 compares the expected number of housing units consumed with the desired investment exposure to house prices over the life-cycle. When young, the individual wants little exposure to house price risk due to the high correlation with the large human wealth and prefers renting to owning. From about ten years before retirement and through most of the retirement phase, the housing investment exceeds the desired housing consumption. This can be implemented in different ways: (i) by buying more housing units and renting out part of those or (ii) by purchasing a house of a size corresponding to the (average) desired housing consumption in the remaining life and investing in the house price related financial assets (quite heavily so just after retirement). Approaching the terminal date, all investments will be liquidated and the individual will prefer renting again. This life-cycle pattern of renting and owning seems consistent with typical real-life behavior. Our model highlights the role of human wealth and the correlation between house prices and labor income in explaining the renting vs. owning decisions over the life-cycle.

4 Comparative statics

In this section we investigate the effects of variations in the value of some of the key parameters and state variables, while keeping other values fixed.

4.1 Bequest motive

In the benchmark case, the individual has no utility from terminal wealth. As an alternative, we now consider a relatively strong bequest motive represented by $\varepsilon = 10$ so that leaving a terminal wealth of $x$ provides a utility ten times as high as consuming $x$ in the final year. The individual then has to reduce consumption at all dates, but only very little as the extra financial wealth can be build up slowly over the lifetime. Figure 6 shows that the extra savings are primarily invested in the housing asset. In the last years before the terminal date, the individual with a bequest motive will also start increasing the savings in the money market account in order to insure the bequest against financial shocks; the money market account is the truly risk-free asset for an individual with a very short investment horizon.

4.2 Risk aversion

The qualitative effects of varying the degree of risk aversion are well-known. With a lower risk aversion, the individual increases the magnitude of the speculative asset demands and decreases the hedging demands, whereas the income-adjustment terms remain unchanged under our assumptions of complete
markets and unconstrained portfolios. The hedging demands are less sensitive to the risk aversion coefficient than the speculative demands. For example, when the risk aversion coefficient $\gamma$ is lowered from 4 to 2, the speculative demand for the housing asset is doubled from 91% to 182% of total wealth and the hedging demand drops from 15% to 10%. Hence, the total demand for the housing asset is much higher for the lower risk aversion. The speculative stock demand is also doubled, but from a low level. For the bond, the speculative demand is negative, whereas the hedging demand is positive. Lowering the risk aversion, the speculative bond demand becomes more negative and the hedging demand less positive, so the total bond demand decreases quite dramatically. Figure 7 illustrates the expected investments over the life-cycle for the benchmark risk aversion of $\gamma = 4$, a lower risk aversion of $\gamma = 2$, and a higher risk aversion of $\gamma = 6$.

With the assumed time-additive power utility preferences, a lower risk aversion goes hand in hand with a higher elasticity of intertemporal substitution and thus a higher variation in consumption over the life-cycle. On average, the larger investments of the less risk averse individual lead to higher wealth and higher consumption throughout life. Figure 8 depicts the expected housing consumption and investment over life for the three values of the relative risk aversion. The housing consumption is initially roughly independent of the degree of risk aversion, but increases much more over life for the least risk-averse individual. The housing consumption just before the terminal date is more than four times higher with a risk aversion of 2 and 30% lower with a risk aversion of 6 compared with the benchmark case in which the risk aversion is 4. Individuals with low risk aversion should buy their own house earlier in life and even seek more exposure to the housing market by buying more housing units and rent out to other consumers or by investing heavily in house-price related financial assets. The reason is the attractive risk-return tradeoff of housing in combination with the correlation structure between house price, income, stocks, and interest rates. Very risk-averse individuals should rent their home over longer periods of their life.

4.3 House price level and volatility

The optimal spending on housing consumption and the optimal fraction of total wealth invested in housing are both independent of the level of house prices, so that the number of housing units consumed and the number of housing units invested in are inversely related to the house price.

An increase in the volatility of house prices $\sigma_H$, for a fixed Sharpe ratio, dampens both the positive speculative demand and the negative income-adjustment term in the fraction of total wealth invested in housing, cf. (3.17), whereas the hedge term remains the same. For very high human-to-total wealth ratios (typically early in life), the income-adjustment term dominates and an increase in $\sigma_H$ will
therefore increase the total fraction of wealth invested in housing. For other values of the human-to-
total wealth ratio, an increased house price volatility leads to a lower fraction of wealth invested in
housing. In our quantitative example, human wealth is initially very high so that an increase in the
house price volatility leads to a higher housing investment in the early years. Due to the attractive
risk premium on the housing asset, the financial wealth is expected to grow faster. Later in life, when
human wealth is a smaller fraction of total wealth, the higher house price volatility will induce a lower
housing investment. However, the lower housing investment will be at a higher expected return, so
that the expected financial wealth will be higher throughout life in the case of a high house price
volatility. Figure 9 illustrates how the house price volatility influences the expected investments in
the stock, the bond, and the housing asset over the life-cycle. The fraction of total wealth invested in
the stock is independent of the house price volatility. The amount invested in the stock consists of a
speculative term which is a constant, positive proportion of total wealth and an income-adjustment
term which is proportional to human wealth. Hence, the increased expected financial wealth induced
by a higher house price volatility leads to higher expected speculative investments in the stocks, but
the quantitative effect is limited when our benchmark parameter values are applied. For the bond,
the house price volatility affects the hedge term via \( g_r/g \), but the quantitative effect is insignificant.
Hence, as for the stock, the main effect of a higher house price volatility on the bond investments
comes from the increase in the expected financial wealth which leads to a more negative speculative
demand and a higher, negative hedge demand, i.e., the expected bond investment drops somewhat.

[Figure 9 about here.]

The house price volatility further affects the optimal consumption strategies via the function \( d_1(\tau) \)
in (3.4), which enters the \( g \)-function that is the key determinant of the consumption-wealth ratios
in (3.8) and (3.9). With the assumed benchmark values of other parameters, \( d_1(\tau) \) and thus the
consumption-wealth ratios are convex second-order polynomials of \( \sigma_H \). By replacing the benchmark
house price volatility of 12% by either 8% or 16%, the consumption-wealth ratios will in both cases
increase, but only very little so that consumption is mainly affected via the wealth. Since a higher house
price volatility induces a higher expected financial wealth, the expected spending on both consumption
goods increase as illustrated in Figure 10. An increase in the house price volatility for a fixed Sharpe
ratio leads to higher expected future house prices, and with the benchmark parameter values the
quantitative effect is significant. For example, the expected unit house price in 20 years is 290 USD
for the benchmark 12% house price volatility, but 229 USD for an 8% volatility and 367 USD for a
16% volatility. When increasing the volatility, the increase in the expected unit house price more than
outweighs the increase in expected spending on housing consumption, so that the number of housing
units occupied by the individual falls as shown by the solid curves in Figure 10.

[Figure 10 about here.]
4.4 Income level, replacement ratio, volatility, and growth rate

An increase in the initial income $Y_0 = y$ leads to a proportional scaling of the entire income stream and thus the human wealth. The income-adjustment term in the portfolio weights (3.15)–(3.17) is positive for the stock and the bond, but negative for the housing asset. An increase in the initial income will therefore cause a higher investment in the stock and the bond, but a lower housing investment as a fraction of the higher total wealth. With our benchmark parameter values, the total effect on the amount invested in housing is negative very early in the considered time period, but soon the wealth increase dominates and also the amount invested in housing starts to increase as a consequence of the increase in initial income. At retirement, the expected bond investment drops to an even more negative value since the negative second income-adjustment term (the term in (3.16) involving $F_r/F$) is larger because of the larger income. Figure 11 illustrates the quantitative impact on the expected investments of a change of the initial income level from $y = 20,000$ USD per year to either 10,000 or 30,000 USD. Of course, a higher income induces higher consumption over the entire life-cycle.

An increase in the income replacement ratio $\Upsilon$ obviously leads to a higher human wealth throughout life. The extra income during retirement allows higher consumption at all ages accompanied by a lower financial wealth. From (3.15)–(3.17), the amounts invested in the risky assets are determined by the total wealth and the human-to-total wealth ratio, which are both higher at all ages. The expected stock investment is higher over the full life-cycle, whereas the expected bond investment is higher before retirement and lower (more negative) in retirement. In contrast, the expected housing investment is lower before retirement and higher in retirement.

Next, consider an increase in the income volatility $\bar{\sigma}_Y$ before retirement. This causes a lower risk-neutral income drift rate $\mu_Q^Y = \bar{\mu}_Y - \bar{\sigma}_Y \lambda_Y$ and thus a lower human wealth, as can be seen formally from the expression (2.14) for the income multiplier and the expression (A.10) for the auxiliary discounting function $A(t,s)$. The lower human wealth has the obvious consequences for consumption over the life-cycle. The investment behavior is further affected by the income-adjustment terms. Figure 12 shows the expected investments over the life-cycle for the three different levels of the income volatility: the benchmark value of 7.5% as well as 5% and 10%. For a higher volatility, the total wealth is always lower. In retirement, the expected stock investment is therefore lower. Before retirement, the lower speculative investment is more than outweighed by the higher income-adjustment term leading to a higher expected stock investment. The result is similar for the bond before retirement, whereas in retirement the lower wealth causes a less negative bond investment. The housing investment drops both before and in retirement. For all three assets, the investments before retirement are quite sensitive

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11With a non-zero instantaneous correlation between the income and the bond, there would be an additional effect through the term with $\rho_{YB}$ in (A.10). For certain parameter constellations, the human wealth might even increase in the income volatility.
to the income volatility due to its direct effect on the large income-adjustment terms.

[Figure 12 about here.]

The dramatic changes in investments at retirement is due to the assumption that the income volatility is constant throughout working life and then suddenly drops to zero, eliminating the large income-adjustment terms involving income volatility. It seems more reasonable to assume that income uncertainty is higher in the younger years, where it is presumably harder to estimate the individual’s skills and potential. To be concrete, Figure 13 considers the case where income volatility is initially 15% and then drops linearly to 0% over the 30 year working life so that the average income volatility is 7.5% as in the benchmark case. The income-adjustment terms in the optimal portfolio weights are now initially much bigger leading to higher investments in the stock and the bond and lower housing investments. With increasing age, the income-adjustment terms become smaller and the expected investments change smoothly to the optimal values immediately after retirement, in contrast to the abrupt changes at retirement seen for a constant income volatility. In addition to the induced age variations in the income-adjustment terms, there is an additional and smaller effect of age-dependent income volatility via its influence on human wealth. The human wealth is lower with the decreasing income volatility than with the constant volatility, and consequently the expected investments are smaller in magnitude throughout the life-cycle. As the quantitative effects of the age-dependent income volatility can be large, it will be important to obtain empirical estimates of the typical variations in income volatility over the life-cycle.

[Figure 13 about here.]

Until now, we have assumed that the expected income growth rate in the entire working life is given by $\bar{\mu} + br$, with $\bar{\mu} = 0.01$ and $b = 0.5$ resulting in a 2% expected growth rate when the interest rate is at its long-term level. Based on data from the Panel Study of Income Dynamics (PSID) in the US, Cocco, Gomes, and Maenhout (2005) have estimated third-order polynomials $a_0 + a_1t + a_2t^2 + a_3t^3$ for the average of the log-income level in each of three different educational groups: no high school, high school, and college education. The relative income growth rate is then $a_1 + 2a_2t + 3a_3t^2$. To encompass both the dependence on age and on the interest rate level, we will assume that the expected income growth rate is $\bar{\mu} + br, + (a_1 - 0.02) + 2a_2t + 3a_3t^2$. The estimates of $a_1$, $a_2$, and $a_3$ are shown in Table 2. In the following, we consider individuals who are initially 25 years old and having a financial wealth of 20,000 USD and an annual income of 20,000 USD. They retire at age 65 and live until age 85. Figure 14 illustrates the expected income as a function of age for a typical individual in each of the three different education groups and, for comparison, an individual with an age-independent expected income growth as assumed hitherto. All three profiles show a significant age-dependence in the income level. Early in life, the income growth is higher than in the benchmark case, in particular for the college graduate. Around age 40-45, the income growth rate is close to zero for all three groups.
and subsequently it becomes slightly negative up to retirement. Next we consider the implications of these differences in labor income for the optimal decisions and the resulting wealth dynamics over the life-cycle. The individuals only differ with respect to their expected income growth in working life; for all other parameters, we assume the benchmark values in Table 1.

[Table 2 about here.]

[Figure 14 about here.]

Figure 15 depicts the expected human, financial, and total wealth of the typical individual in each of the three educational groups. Due to the differences in the income growth rate, the human wealth of the college graduate will at any age exceed that of the high school educated, which again is higher than the human wealth of the uneducated. Hence, the college graduate can consume more than the others throughout life, which in the early years results in a lower financial wealth, but around age 40 he is starting to build up more financial wealth than the others in order to finance the higher consumption in the rest of the life. Concerning the investments, the optimal fractions of total wealth in (3.15)–(3.17) as a function of the human-to-total wealth ratio are the same for all three individuals, except for small differences in the \( F_r/F \) ratio entering the fraction of wealth invested in the bond. Differences in the amounts invested in the assets are thus mostly due to differences in the magnitude of total wealth and the human-to-total wealth ratio across the three educational groups. The expected amounts invested in the risky assets by the three individuals can be seen in Figure 16. Due to the higher human wealth and total wealth, the investments of the college graduate are generally of a larger magnitude than for the two less educated individuals. The higher speculative demand of the college graduate comes with a higher income-adjustment term. For the stock and the bond, the terms work in the same direction, but for the house the higher speculative demand is in the early years more than off-set by the more negative income-adjustment term. Hence, the college graduate is expected to invest less in the housing asset early in life relative to the other individuals. This is also apparent from Figure 17 which compares the expected housing consumption and investment of the three individuals. The college graduate will, in expectation, consume more housing services throughout life. The less educated will sooner reach the point in time at which the desired housing investment is as big as the desired housing consumption. One interpretation is that, other things equal, individuals with higher education (or rather: higher human wealth) should engage in home ownership later in life than others. The main driver behind this result is the income-adjustment terms of the optimal investments and a significant positive correlation between house prices and labor income. From around age 50, the difference between housing investment and housing consumption is highest for the college graduate, so mature and highly educated individuals should be the most active when it comes to buying REITs and house price linked financial contracts or the most active buying additional real estate and renting out part of that. This behavior is mainly driven by the attractiveness of housing as an asset class.

[Figure 15 about here.]
4.5 Correlations

The instantaneous correlation structure between the prices of the stock, the bond, and the housing asset is central for the speculative demands as reflected by the constants $\xi_B$, $\xi_S$, and $\xi_I$ defined in (B.14)–(B.16). And the correlation structure between the three asset price and the labor income is a key determinant of the quantitatively sizeable income-adjustment terms in the optimal investment strategies. Our assumption of complete markets, which is necessary to derive closed-form solutions, makes it impossible to vary one pairwise correlation parameter without varying at least one other correlation parameter. This limitation can lead to counter-intuitive results. For example, it would be interesting to vary the correlation $\rho_{YH}$ between the labor income and the house price, since this correlation appears to be driving force behind the large income-adjustment term in the investment demand for the housing asset causing large variations over the life-cycle. Increasing $\rho_{YH}$, the human wealth will resemble the housing asset more, so the income-adjustment term subtracted from the housing investment should be of an even bigger magnitude leading to a lower investment demand for housing. If we fix $\rho_{SB}$, $\rho_{YB}$, and $\rho_{YS}$ at their benchmark values of 0, it follows from (2.12) that

$$\rho_{YH} = \sqrt{1 - \rho_{HB}^2 - \rho_{HS}^2},$$

(4.1)

and it follows from (A.5) that the parameter $\zeta_I$ in the income-adjustment term of the housing investment is given by

$$\zeta_I = \frac{1}{\sqrt{1 - \rho_{HB}^2 - \rho_{HS}^2}}.$$

If we want a higher $\rho_{YH}$, we have to lower (the absolute value of) either $\rho_{HB}$ or $\rho_{HS}$ to ensure perfect spanning. This leads to an income-adjustment term which is lower in magnitude, going against the intuition explained above. The change in $\rho_{HB}$ or $\rho_{HS}$ will further affect both the speculative demands for all assets and the income-adjustment terms in the stock and bond demands, as well as the valuation of human capital through the “income Sharpe ratio” $\lambda_Y$ in (A.2). While $\rho_{HB}$ may be difficult to change, the investor can to some extent affect $\rho_{HS}$ by tilting the stock portfolio more or less against REITs or other stocks highly correlated with house prices. Figure 18 shows how the expected investments over the life-cycle are affected when the value of $\rho_{HY}$ is changed from the benchmark value of 0.5723 to either 0.5 or 0.65 and $\rho_{HS}$ is simultaneously varied so that (4.1) continues to hold (from 0.5 to either 0.5723 or 0.3937). Indeed, with a higher house-income correlation, the housing investment is initially higher due to the smaller subtracted income-adjustment term, but the speculative housing demand is lower than in the benchmark case (due to the derived effects from $\rho_{HS}$) and will lead to a smaller expected housing investment later in life when human wealth falls.
In order to give a better impression of the isolated effects of varying the house-income correlation, we extend the income dynamics in (2.8) to

\[ dY_t = Y_t \left[ \mu_Y(r_t, t) dt + \sigma_Y(t) (\rho_{YB} dW_{rt} + \tilde{\rho}_{YS} dW_{St} + \tilde{\rho}_{YH} dW_{Ht} + \hat{\rho}_Y dW_{Yt}) \right], \]

where \( W_Y = (W_{Yt}) \) is a standard Brownian motion independent of \( W_r, W_S, \) and \( W_H, \) and where \( \hat{\rho}_Y = \sqrt{1 - \rho_{YB}^2 - \rho_{YS}^2 - \rho_{YH}^2} \) (in our standard complete market model, \( \hat{\rho}_Y = 0 \) and thus \( \hat{\rho}_{YH} = \sqrt{1 - \rho_{YB}^2 - \rho_{YS}^2} \)). If we keep \( \rho_{YB} = \rho_{YS} = \rho_{SB} = 0, \) we have \( \hat{\rho}_{YH} = \rho_{YH} \hat{\rho}_H \) and \( \hat{\rho}_Y = \sqrt{1 - \rho_{YH}^2}. \) Here, \( \hat{\rho}_{YH} \) determines the degree of spanning of the income stream. We can vary \( \rho_{YH} \) without simultaneously varying \( \rho_{HH} \) or \( \rho_{HS}. \) Our closed-form solution is not valid when the income is not perfectly spanned. However, the results presented in Bick, Kraft, and Munk (2009) for a similar, slightly simpler, model indicate that when the income is closed to being spanned, the strategy derived assuming perfect spanning will be near-optimal, in the sense that the investor will suffer only a small certainty-equivalent loss by following this closed-form sub-optimal strategy instead of the unknown optimal strategy.\(^{12}\) Figure 19 illustrates the expected investments in the risky assets for the same three values of the house-income correlation as considered above: 0.5 (low), 0.5723 (benchmark), and 0.65 (high), but now we fix the house-stock correlation at 0.3937. The income is then perfectly spanned only when \( \rho_{HY} = 0.65, \) not for the two lower house-income correlations. The speculative demands as a fraction of total wealth are the same in all three cases. The human wealth and thus the total wealth are also affected (decreasing in the house-income correlation), which again affects the amounts invested in the assets for speculative reasons. Still, the differences in the amounts invested are mainly stemming from the income-adjustment terms. In line with intuition, the housing investment is now decreasing in the house-income correlation throughout life. For \( \rho_{HY} = 0.5, \) the investment position in housing units exceeds the housing consumption from the beginning, whereas for \( \rho_{HY} = 0.5723 \) and \( \rho_{HY} = 0.65 \) housing investment is expected to exceed housing consumption around 8 and 15 years, respectively, into the future. Other things equal, individuals having a labor income more highly correlated with house prices should wait longer before buying their first home. The positive income-adjustment terms, and hence the total investments, for the stock and the bond increase in magnitude for higher values of the house-income correlation.

\[ \text{[Figure 19 about here.]} \]

\(^{12}\)When the income is not perfectly spanned, the human wealth cannot be uniquely valued by no arbitrage arguments. The “income Sharpe ratio” \( \lambda_Y \) in (A.2) determining the risk-neutral income drift will then involve the market price of risk associated with the shock \( W_Y, \) and this market price of risk is specific to the individual and can only be deduced from the (unknown) truly optimal consumption and investment strategy in the incomplete market. Applying our complete markets solution in the incomplete market setting, we basically assume that this investor-specific market price of risk associated with \( W_Y \) is zero.
5 Robustness to infrequent housing decisions

The model of Section 2 was designed with the purpose of allowing for closed-form solutions for the optimal housing, consumption, and investment strategy as found in Section 3. One questionable assumption of the model is that the individual is able to adjust the housing consumption and investment positions continuously over time. Clearly, it is practically inefficient to adjust the physical ownership of housing units continuously due to explicit and implicit transaction costs. Continuous adjustment of the investment in REITs may be a reasonable approximation to real life, but if REITs linked to the house prices of interest to the investor are not traded, the individual cannot continuously adjust the housing investment position. If changes in the renting position are costly, continuous adjustment of the housing consumption position will also be inefficient.

In this section we relax the assumption and consider strategies that involve only infrequent adjustments of the housing consumption and/or the investment position in housing. Clearly, the expected utility generated by any such constrained strategy, denoted by $\hat{J}(t, x, r, h, y)$, is smaller than the expected utility following the optimal unconstrained strategy, i.e., $J(t, x, r, h, y)$ from Theorem 3.1. We measure the economic importance of the imposed constraint by the percentage decrease in initial financial wealth and labor income (and thus in initial total wealth) necessary to bring the optimal expected utility down to the expected utility obtained with limited flexibility:

$$J(t, x[1 - \ell], r, h, y[1 - \ell]) = \hat{J}(t, x, r, h, y) = J(t, x, r, h, y).$$  \hfill (5.1)

We can interpret $\ell$ as the percentage wealth loss the individual would incur if he were restricted to the constrained strategy or the percentage of wealth the individual is willing to sacrifice to avoid the constraints. Due to the form of the value function in (3.1), we get

$$\ell = 1 - \left( \frac{\hat{J}(t, x, r, h, y)}{J(t, x, r, h, y)} \right)^{\frac{1}{\gamma}}.$$  \hfill (5.2)

As we will show below, the wealth-equivalent losses associated with the strategies featuring inflexible housing decisions are small, which confirms the relevance of our closed-form solutions in realistic settings.

5.1 Constant housing consumption: an explicit solution

First, we discuss a case where we can find an explicit solution to the problem with limited flexibility. We consider a power utility investor who gets no utility from terminal wealth ($\varepsilon = 0$) and is restricted to a fixed consumption of housing services given by the constant $\bar{\varphi}_C$. Due to (2.18) we can assume that the individual satisfies her housing consumption through renting and does not have direct ownership of housing, i.e., $\varphi_{ot} \equiv 0$. The individual can then obtain the desired exposure to house price risk by investing in REITs as captured by $\varphi_{Rt}$. Alternatively, we can ignore REITs and let the individual continuously adjust the direct ownership of houses, but then the housing units rented would have to
be adjusted continuously so that the sum $\varphi_{rt} + \varphi_{ct}$ equals the deterministic total housing consumption $\bar{\varphi}_C$. In Appendix C we demonstrate the following result (the superscript “dc” indicates deterministic consumption):

**Theorem 5.1** Assume $\varepsilon = 0$ and a fixed housing consumption $\bar{\varphi}_C$. The value function is given by

$$J^{dc}(t, x, h, y) = \frac{1}{1 - \tilde{\gamma}} g^{dc}(t, y)(x + yF(t, r) - \nu h\tilde{F}(t))^{1-\tilde{\gamma}},$$

where $\tilde{\gamma} = 1 - \beta(1 - \gamma) \in (1, \gamma)$, $F$ is defined in (2.14),

$$\tilde{F}(t) = \bar{\varphi}_C B_u(T - t),$$

$$g^{dc}(t, y) = \beta^\dagger \phi^{1-\tilde{\gamma}} \int_t^T e^{-D_t(u-t) - \frac{yF(u)}{yF(t)}} B_s(u-t) \, du,$$

and $D_t$ is given by (3.3) with $\tilde{\gamma}$ replacing $\gamma$. The optimal fractions of tangible wealth invested in the stock and the bond are

$$\pi_S = \frac{1}{\tilde{\gamma}} \frac{\xi_S x + yF - \nu h\tilde{F}}{\sigma_S x - yF},$$

$$\pi_B = \frac{1}{\tilde{\gamma}} \frac{\xi_B x + yF - \nu h\tilde{F}}{\sigma_B x - yF} - \left(\frac{\sigma_B yF}{\sigma_B x} - \frac{\sigma_x yF}{\sigma_B x}\right) - \frac{\sigma_x g^{dc} x + yF - \nu h\tilde{F}}{\sigma_B g^{dc} x},$$

respectively, while the optimal number of housing units invested in is

$$\varphi_I = \frac{1}{\tilde{\gamma}} \frac{\xi_I x + yF - \nu h\tilde{F}}{\bar{\varphi}_C h} - \frac{\sigma y(t)\xi_I yF}{\sigma_H h} + \nu \tilde{F}$$

and the optimal consumption of the perishable good is

$$c = \left(\int_t^T e^{-D_t(u-t) - \frac{yF(u)}{yF(t)}} B_s(u-t) \, du\right)^{-1} \left(x + yF - \nu h\tilde{F}\right).$$

In all these expressions, $y$ is to be replaced by $Y_{\tilde{T}}$ when $t \in [\tilde{T}, T]$.

Relative to the form of the value function in Section 3 there are three differences. First, the relevant $g$-function depends on $r$, which captures financial investment opportunities, but not on the relative price of consumption goods $h$ since the individual cannot control the consumption of housing. Second, note that the present value of all future renting expenses is given by the risk-neutral expectation\(^{13}\)

$$E_t^Q \left[\int_t^T e^{-\int_t^r r_u \, du} \bar{\varphi}_C \nu H_s \, ds\right] = \nu H_t \bar{\varphi}_C \int_t^T E_t^Q \left[e^{-\int_t^{s} r_u \, du} \frac{H_s}{H_t}\right] \, ds = \nu H_t \bar{\varphi}_C \int_t^T e^{-\nu(s-t)} \, ds = \nu H_t \tilde{F}(t).$$

The total initial wealth at the disposal of the individual is thus the initial tangible wealth plus the human capital minus the present value of future rents, $x + yF(t, r) - \nu h\tilde{F}(t)$. Third, compared to the

\(^{13}\)The risk-neutral drift of house prices is the real-world drift $r_t + \lambda_H \sigma_H - \nu^{imp}$ minus the risk premium $\lambda_H \sigma_H$, which equals $r_t - \nu$.  

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separation (3.1) in the previous section, $\tilde{\gamma}$ has replaced $\gamma$ in the exponents of $g$ and the total present wealth.\textsuperscript{14}

Compared to the optimal investment strategies derived in Section 3, the smaller effective relative risk aversion and the smaller “disposable wealth” work in opposite directions, and the net effect depends on precise parameter values. The adjustment for the risk exposure of human wealth is the same as before. Concerning the demand for housing investment, the last term in (5.8) ensures that when the present value of future renting costs increases, the tangible wealth increases by at least the same amount. Hence, the individual will always have sufficient wealth to pay the rent. The optimal perishable consumption is a time- and state-dependent fraction of the “disposable wealth”, a fraction that depends on the constant level of housing consumption. The perishable consumption is decreasing in the housing consumption level as expected.

Theorem 5.1 gives the value function for any constant level of housing consumption, $\varphi_C$. We can perform an initial (time $t = 0$) optimization over $\varphi_C$ to find the optimal constant consumption of housing services. The first-order condition for the maximization of $J(0, x, r, h, y)$ with respect to $\varphi_C$ implies

$$\frac{\tilde{\gamma}}{1 - \tilde{\gamma}} \frac{\partial g^{dc}(0, r)}{\partial \varphi_C}(x + yF(0, r)) = \nu hB_p(T) \left( g^{dc}(0, r) + \frac{\tilde{\gamma}}{1 - \tilde{\gamma}} \frac{\partial g^{dc}(0, r)}{\partial \varphi_C} \varphi_C \right),$$

(5.11)

and since $\frac{\partial g^{dc}(0, r)}{\partial \varphi_C} \varphi_C = (1 - \gamma/\tilde{\gamma})g^{dc}(0, r)$, we get an optimal housing consumption of

$$\varphi_C = (1 - \beta) \frac{X_0 + Y_0F(0, r_0)}{\nu H_0B_p(T)},$$

(5.12)

i.e., so that initially the expenditure on house consumption is a fraction $(1 - \beta)/B_p(T)$ of total wealth. With our benchmark parameters, a financial wealth of $X_0 = 20,000$ USD, an initial annual income of $Y_0 = 20,000$ USD, and a unit house price of $H_0 = 250$ USD, we get a housing consumption of $\varphi_C \approx 490$ units.

The wealth-equivalent utility loss an individual suffers by having to stick to a constant consumption of housing is given by (5.2) with $\tilde{J}$ replaced by $J^{dc}$. With the benchmark parameters the loss from applying the optimal constant level of housing consumption instead of the optimal fully flexible housing consumption strategy is 0.24% of total initial wealth, i.e., an individual with full flexibility in housing consumption decisions is willing to give up only 0.24% of his total wealth in order to avoid being restricted to consuming the same level of housing services across time and states.\textsuperscript{15} In our benchmark case, initial total wealth is around 560,000 USD, so the loss is less than 1,400 USD. Figure 20 illustrates how the utility loss varies with the chosen fixed housing consumption. Note that the curve is quite

\textsuperscript{14}The intuition is as follows. The consumption of the two goods enter the utility function via the term $(c^a\varphi_C^{1-a})^{1-\gamma} = (c^a\varphi_C^{1-a})^{1-\gamma}$. When we can freely choose $c$ and $\varphi_C$, $\gamma$ will be the effective relative risk aversion, and is therefore the relevant parameter in the exponents of the separation (3.1). Now the assumption is that the individual can only choose how much of the perishable good to consume, and since $(c^a\varphi_C^{1-a})^{1-\gamma} = \varphi_C^{(1-\beta)(1-\gamma)}c^{(1-\gamma)} = \varphi_C^{(1-\beta)(1-\gamma)}c^{1-\tilde{\gamma}}$, we can see that $\tilde{\gamma}$ now plays the same role as $\gamma$ in the preceding section, i.e., $\tilde{\gamma}$ is the effective relative risk aversion.

\textsuperscript{15}The minimum loss is 1.73% for a relative risk aversion of 2 and 0.41% for a relative risk aversion of 6.
flat around the optimum and since the expected units of housing consumed according to Figure 1 does not fluctuate wildly over life, it should be expected that little can be gained by replacing the optimal constant $\bar{\varphi}_C$ by any deterministic consumption schedule. More surprisingly, the very small wealth loss associated with a constant housing consumption suggests that little can be gained by letting the consumption of housing services be state-dependent. The individual can almost completely compensate for the inflexibility in housing consumption via appropriate adjustments of the perishable consumption and the investments in stocks, bonds, and REITs.

[Figure 20 about here.]

5.2 Infrequent housing adjustments: Monte Carlo results

Next, we implement a Monte Carlo simulation procedure to investigate the welfare loss suffered by an individual restricted to infrequent adjustments of (a) housing consumption $\varphi_{Ct}$, (b) housing investment $\varphi_{It}$, or (c) both $\varphi_{Ct}$ and $\varphi_{It}$. We experiment with adjustment frequencies of 2 and 5 years. All other controls are adjusted at every time step used in the Monte Carlo simulation of the state variables, i.e., 250 times a year (roughly once per trading day).\(^\text{16}\) Whenever a control variable is adjusted, it is reset to the optimal value stated in Theorem 3.1 using the simulated values of wealth, the interest rate, the house price, and the labor income rate. For each simulation path we compute the life-time utility of consumption and terminal wealth and approximate expected utility by averaging over 10,000 paths. The associated welfare loss is then computed from (5.2).\(^\text{17}\)

Table 3 reports the wealth-equivalent loss for both adjustment frequencies assuming the benchmark parameter values. For the initial income level of 20,000 USD assumed in our other examples, we see that the loss is relatively small (below 2%) even when both housing consumption and housing investments are reset only every 2 or 5 years. In our simulation experiment, the individual restricted to infrequent housing adjustments is required to follow the investment strategy in the stock and the bond that is optimal in a situation with continuous housing adjustments. Between adjustment dates of the housing investment position, the individual has a suboptimal exposure to all risks due to the suboptimal housing investment and he could partly compensate for that by choosing slightly different stock and bond positions.\(^\text{18}\) Therefore the losses reported in the table exaggerate the true losses. Consistent with intuition, we see from the table that the loss increases with the period between adjustments. Apparently, it has very little value to be able to adjust the housing consumption position frequently, in line with the results of Section 5.1. Frequent adjustments of the housing investment

\(^{\text{16}}\)The results are similar for 12 time steps per year corresponding to monthly rebalancing.

\(^{\text{17}}\)For the purpose of computing the loss, we also compute the expected utility generated by the optimal unconstrained strategy by Monte Carlo simulation using the same set of random numbers as was used to compute the utility of the constrained strategy. This will reduce any bias in the loss due to the simulation procedure.

\(^{\text{18}}\)The truly optimal stock and bond positions in the situation with infrequent housing adjustments can only be computed by numerical dynamic programming techniques.
position are more valuable, which indicates that a well-functioning market for REITs or other financial contracts facilitating the hedging of house price risks can have a non-negligible welfare effect. The loss due to infrequent rebalancing of the housing investment position is considerably smaller when the initial income (and thus the entire human wealth) is reduced by 50%, which is again due to the close relation between the optimal housing investment and the magnitude of human wealth. REITs are therefore more valuable for individuals having a large human wealth. The relatively small losses due to infrequent housing transactions also suggest that moderate transaction costs will be of minor importance to the welfare of the individual in the sense that an investor resetting his housing positions (and paying the associated transaction costs) infrequently to the positions derived in Theorem 3.1 will obtain almost the same utility as if he could continuously adjust the housing positions at zero transaction costs.

[Table 3 about here.]

6 Conclusion

We have provided explicit solutions to quite complicated life-cycle utility maximization problems having many important and realistic features. The explicit consumption and investment strategies are simple and intuitive and have been discussed and illustrated in the paper.

For a calibrated version of the model we find, among other things, that the fairly high correlation between labor income and house prices imply much larger life-cycle variations in the desired exposure to house price risks than in the exposure to the stock and bond markets. Young individuals want little or even negative exposure to house price risk as they are highly exposed to the labor income risk, which is positively correlated with house prices. Hence, young individuals rent their home and may even short financial assets linked to house prices. Later in life, when human wealth declines, the desired housing investment increases and will eventually reach and exceed the desired housing consumption, suggesting that the individual should buy his home and maybe even obtain a higher exposure to house prices, either by buying additional housing units and renting them out or by taking long positions in house price linked financial assets. In the final years, the desired housing investment again falls below the desired housing consumption, indicating a shift back to home rental.

Our comparative statics contain numerous interesting results. For example, very risk-averse individuals are more inclined to rent than to own their home. Individuals with higher human wealth should wait longer before engaging in home ownership, and later in life they should have a higher exposure to the housing market either through long positions in house price linked financial assets or by buying and renting out additional residential property. While our model involves continuous adjustments of the consumption of housing services and the exposure of wealth to house price risk, we have demonstrated that the derived strategies are still very useful if the housing positions are only reset infrequently. Our results suggest that markets for REITs or other financial contracts linked to house prices will lead to

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non-negligible improvements of welfare.
A Proof of Theorem 2.1 (human capital)

(a) Human capital is defined in (2.9). In retirement, there is no uncertainty about future income so the human capital becomes

\[ L_t = E_t^Q \left[ \int_t^T e^{-\int_t^s r_u \, du} Y_t \, ds \right] = YY_T \int_t^T E^Q_t [e^{-\int_t^s r_u \, du}] \, ds = \int_t^T e^{-a(s-T)} - B_s(s-T) r_u \, ds. \]

Before retirement, i.e., for \( t \leq \bar{T} \), the human capital can be split in two:

\[
\begin{align*}
L_t &= E_t^Q \left[ \int_t^\bar{T} e^{-\int_t^s r_u \, du} Y_s \, ds + \int_\bar{T}^T e^{-\int_t^s r_u \, du} Y_s \, ds \right] \\
&= E_t^Q \left[ \int_t^\bar{T} e^{-\int_t^s r_u \, du} Y_s \, ds \right] + E_t^Q \left[ e^{-\int_t^\bar{T} r_u \, du} Y_s \, ds + \int_\bar{T}^T e^{-\int_t^T r_u \, du} \right] \left[ E_t^Q \left[ e^{-\int_t^\bar{T} r_u \, du} Y_s \, ds \right] \right]
\end{align*}
\]

\[
\begin{align*}
&= E_t^Q \left[ \int_t^\bar{T} e^{-\int_t^s r_u \, du} Y_s \, ds \right] + \mathbb{E}_t^Q \left[ e^{-\int_t^\bar{T} r_u \, du} Y_s \, ds + \int_\bar{T}^T e^{-a(s-\bar{T})-B_s(s-\bar{T}) r_u \, ds} \right].
\end{align*}
\]

To proceed, we have to identify the risk-neutral dynamics of the income and the short-term interest rate. The process \( W = (W_r, W_S, W_H)^T \) is a 3-dimensional standard Brownian motion under the real-world probability measure. With \( \lambda = (\lambda_B, \lambda_S, \lambda_H)^T \) and

\[
\Sigma = \begin{pmatrix}
1 & 0 & 0 \\
\rho_{SB} & \sqrt{1 - \rho_{SB}^2} & 0 \\
\rho_{HB} & \rho_{HS} & \hat{\rho}_H
\end{pmatrix},
\]

the market price of risk vector is \( \Sigma^{-1} \lambda \). Therefore, the process \( W_t^Q = (W_r^Q, W_S^Q, W_H^Q)^T \) defined by initial value zero and

\[
dW_t^Q = dW_t + \Sigma^{-1} \lambda \, dt
\]

is a standard Brownian motion under the risk-neutral measure \( Q \). Let \( \tilde{\mu}_Y = (\tilde{\rho}_Y B, \tilde{\rho}_Y S, \tilde{\rho}_Y H)^T \) and impose the specification of \( \mu_Y \) and \( \sigma_Y \) in (2.10). The risk-neutral income dynamics is then

\[
\begin{align*}
dY_t &= Y_t \left[ (\tilde{\mu}_Y(t) + br_t) \, dt + \sigma_Y(t) \tilde{\mu}_Y(t)^\top dW_t \right] \\
&= Y_t \left[ (\tilde{\mu}_Y(t) - \sigma_Y(t) \lambda_Y + br_t) \, dt + \sigma_Y(t) \tilde{\mu}_Y(t)^\top dW_t \right] \\
&= Y_t \left[ \left( \mu_Y^Q(t) + br_t \right) \, dt + \sigma_Y(t) \tilde{\mu}_Y(t)^\top dW_t \right],
\end{align*}
\]

where \( \mu_Y^Q(t) = \tilde{\mu}_Y(t) - \sigma_Y(t) \lambda_Y \),

\[
\lambda_Y = \tilde{\rho}_Y \Sigma^{-1} \lambda = \zeta_B \lambda_B + \zeta_S \lambda_S + \zeta_I \lambda_H, \\
\zeta_B = \rho_{YB} - \rho_{SB} \tilde{\rho}_H \tilde{\rho}_H - \tilde{\rho}_Y \tilde{\rho}_H, \\
\zeta_S = \frac{\tilde{\rho}_H \tilde{\rho}_HS - \tilde{\rho}_Y \tilde{\rho}_H}{\tilde{\rho}_H \sqrt{1 - \rho_{SB}^2}}, \\
\zeta_I = \frac{\tilde{\rho}_Y H}{\tilde{\rho}_H}.
\]

(A.5)
The income rate at a future date $\tau \in [t, \bar{T}]$ is thus

$$
Y_\tau = Y_t \exp \left\{ \int_t^\tau \left( \mu^Q_Y(u) - \frac{1}{2} \sigma^2_Y(u) + br_u \right) du + \int_t^\tau \sigma_Y(u) \rho^\top_Y \, dW^Q_u \right\}.
$$

The risk-neutral dynamics of the short rate is

$$
dr_t = (\kappa [\bar{r} - r_t] + \sigma_r \lambda_B) \, dt - \sigma_r \, dW^Q_r = \kappa \left[ \bar{r}^Q - r_t \right] \, dt - \sigma_r \, dW^Q_r,
$$

where $\bar{r}^Q = \bar{r} + \sigma_r \lambda_B / \kappa$. As a consequence,

$$
r_u = r_t e^{-\kappa(u-t)} + \bar{r}^Q \left( 1 - e^{-\kappa(u-t)} \right) - \int_t^u \sigma_r e^{-\kappa(u-u')} \, dW^Q_u
$$

and

$$
\int_t^\tau r_u \, du = r_t B_\kappa (\tau - t) + \bar{r}^Q (\tau - t - B_\kappa (\tau - t)) - \int_t^\tau \sigma_r B_\kappa (u) \, du.
$$

First, for any $s \in [t, \bar{T}]$, we compute

$$
E^Q_t \left[ e^{-\int_t^\tau r_u \, du} Y_s \right] = Y_t e^{\int_t^\tau \left( \mu^Q_Y(u) - \frac{1}{2} \sigma^2_Y(u) \right) du + \int_t^\tau \sigma_Y(u) \rho^\top_Y \, dW^Q_u}
$$

$$
= Y_t e^{\int_t^\tau \left( \mu^Q_Y(u) - \frac{1}{2} \sigma^2_Y(u) \right) du + \int_t^\tau \sigma_Y(u) \rho^\top_Y \, dW^Q_u}
$$

$$
\times \left\{ \int_t^s \left( \alpha_Y(u) - \mu^Q_Y(u) \right) du + \int_s^\tau \sigma_Y(u) \rho^\top_Y \, dW^Q_u \right\}
$$

$$
= Y_t e^{-A(t, s)} e^{\int_t^s \rho^\top_Y \, dW^Q_u}
$$

where $\bar{c} \equiv (1, 0, 0)^\top$ and

$$
A(t, s) = -\int_t^s \mu^Q_Y(u) du - (1 - b) \sigma_r \rho_Y B \int_s^\tau \sigma_Y(u) B_\kappa (s - u) du + (1 - b) \bar{r}^Q (s - t - B_\kappa (s - t))
$$

$$
- \frac{1}{2} (1 - b)^2 \sigma^2_Y \frac{\kappa}{2} \left( s - t - B_\kappa (s - t) \right)^2.
$$

The first equality above follows from (A.6), the second from (A.9), the third from properties of the standard Brownian motion, and the fourth from integrals computed in Appendix D.

Next, for any $s \in [\bar{T}, T]$, we compute

$$
E^Q_t \left[ e^{-\int_t^\tau r_u \, du} Y_T e^{-a(s - \bar{T}) - B_\kappa (s - \bar{T}) r_\bar{T}} \right]
$$

$$
= Y_t e^{\int_t^\tau \left( \mu^Q_Y(u) - \frac{1}{2} \sigma^2_Y(u) \right) du + \int_t^\tau \sigma_Y(u) \rho^\top_Y \, dW^Q_u}
$$

$$
\times e^{-a(s - \bar{T}) - B_\kappa (s - \bar{T}) r_\bar{T} - \int_t^\tau (1 - b) \sigma_Y(u) B_\kappa (u) du + \int_t^\tau \sigma_Y(u) \rho^\top_Y \, dW^Q_u}
$$

$$
= Y_t e^{-A(t, s) - \int_t^s \rho^\top_Y \, dW^Q_u}
$$

where $c = (1, 0, 0)^\top$ and

$$
A(t, s) = -\int_t^s \mu^Q_Y(u) du - (1 - b) \sigma_r \rho_Y B \int_s^\tau \sigma_Y(u) B_\kappa (s - u) du + (1 - b) \bar{r}^Q (s - t - B_\kappa (s - t))
$$

$$
- \frac{1}{2} (1 - b)^2 \sigma^2_Y \frac{\kappa}{2} \left( s - t - B_\kappa (s - t) \right)^2.
$$

The first equality above follows from (A.6), the second from (A.9), the third from properties of the standard Brownian motion, and the fourth from integrals computed in Appendix D.
where
\[
\tilde{A}(t, s) = A(t, \bar{T}) + a(s - \bar{T}) - B_\kappa(s - \bar{T}) E\left[\sigma_\tau^2(1 - b)\frac{1}{\kappa} \left( B_\kappa(\bar{T} - t) - B_{2\kappa}(\bar{T} - t) \right) - \bar{r}^2\kappa B_\kappa(\bar{T} - t) \right] + \frac{1}{2} \sigma_\tau^2 B_{2\kappa}(\bar{T} - t) B_\kappa(s - \bar{T}) + \sigma_r \rho B \int_{t}^{\bar{T}} \sigma_Y(u) e^{-\kappa(\bar{T} - u)} du. \tag{A.11}
\]

The first equality follows from (A.6), the second from (A.8) and (A.9), and the third from properties of the standard Brownian motion as well as results from Appendix D.

The expression for human capital stated in (2.13) now follows from integration of the risk-neutral expectations computed above.

For use in later proofs, we note that for \( t < \bar{T} \), the function \( F(t, r) \) will satisfy the partial differential equation
\[
\frac{\partial F}{\partial t} + F_r \left( \kappa \left[ \bar{r} - r \right] - \rho \sigma_Y \sigma_r \sigma_Y(t) \right) + \frac{1}{2} F_{rr} \sigma_r^2 - F \left( (1 - b)r - \mu_Y^t(t) \right) + 1 = 0, \tag{A.12}
\]
as can be verified by computing the relevant derivatives of \( F \) and substituting them into (A.12). \( F \) satisfies the boundary condition
\[
F(\bar{T}, r) = \Upsilon \int_{\bar{T}}^{T} e^{-a(s - \bar{T}) - B_\kappa(s - \bar{T})r} ds,
\]
which follows from the general specification of \( F \) and the facts that \( \tilde{A}(\bar{T}, \bar{T}, s) = a(s - \bar{T}) \) and \( B_{2\kappa}(\bar{T} - \bar{T}) = 0 \).

(b) Note that we are now considering the behavior of the income and the interest rate under the real-world probability measure. Analogously to (A.6), (A.8), and (A.9), we have
\[
Y_t = Y_0 \exp \left\{ \int_{0}^{t} \left( \mu_Y(u) - \frac{1}{2} \sigma_Y(u)^2 + br_u \right) du + \int_{0}^{t} \sigma_Y(u) \rho_r^\top du \right\}, \quad t \leq \bar{T}, \tag{A.13}
\]
\[
r_t = r_0 e^{-\kappa t} + \bar{r} \left( 1 - e^{-\kappa t} \right) - \int_{0}^{t} \sigma_r e^{-\kappa(t-u)} dW_{ru}, \tag{A.14}
\]
\[
\int_{0}^{t} r_u du = r_0 B_\kappa(t) + \bar{r} (t - B_\kappa(t)) - \int_{0}^{t} \sigma_r B_\kappa(t-u) dW_{ru}. \tag{A.15}
\]
By substituting (A.15) into (A.13) and applying basic properties of stochastic integrals, it can be shown that the expected future income rate is given by (2.15) and that
\[
Y_t = E_0[Y_t e^{b_1 t r_1 + \frac{b_1}{2} \int_{0}^{t} \sigma_Y(u)^2 du + \frac{1}{2} \int_{0}^{t} \sigma_Y(u) \rho_r^\top du} B_\kappa(t-u) dW_{ru} \times e^{-\frac{1}{2} \int_{0}^{t} \sigma_Y(u)^2 du + \frac{1}{2} \int_{0}^{t} \sigma_Y(u) \rho_r^\top du} dW_u, \quad t \leq \bar{T}. \tag{A.16}
\]
For \( t < \bar{T} \), the expectation at time 0 of the human capital at time \( t \) is thus
\[
E_0[L_t] = E_0[Y_t F(t, r_t)]
= E_0[Y_t] E_0 \left[ e^{b_1 t r_1 + \frac{b_1}{2} \int_{0}^{t} \sigma_Y(u)^2 du + \frac{1}{2} \int_{0}^{t} \sigma_Y(u) \rho_r^\top du} B_\kappa(t-u) dW_{ru} \times e^{-\frac{1}{2} \int_{0}^{t} \sigma_Y(u)^2 du + \frac{1}{2} \int_{0}^{t} \sigma_Y(u) \rho_r^\top du} dW_u \right.
\times \left( \int_{t}^{\bar{T}} e^{-\frac{1}{2} \int_{s}^{\bar{T}} \sigma_Y(s) \rho_r^\top du} \right) ds + \Upsilon \int_{\bar{T}}^{T} e^{-\frac{1}{2} \int_{s}^{T} \sigma_Y(s) \rho_r^\top du} \right] ds.
\]
After substitution of (A.14), tedious computations involving basic properties of stochastic integrals lead to (2.16), where

\[
\begin{align*}
   f_1(t, s, \tau) &= (B_\kappa(s-t) - b\kappa(\tau - t))(r_0e^{-\kappa t} + \tau(1 - e^{-\kappa t}) - \frac{1}{2}\sigma^2[B_\kappa(s-t) - b\kappa(\tau - t)]B_\kappa(t) \\
   &+ b\sigma^2\int_0^t e^{-\kappa(t-u)}B_\kappa(t-u)du - \sigma_\tau\rho_\tau B \int_0^t \sigma_Y(u)e^{-\kappa(t-u)}du).
\end{align*}
\]

For \( t \in [\bar{T}, T] \), we first substitute

\[
   r_t = r_\bar{T}e^{-\kappa(t-\bar{T})} + \tilde{\tau}(1 - e^{-\kappa(t-\bar{T})}) - \int_\bar{T}^T \sigma_\tau e^{-\kappa(t-u)}d\tau
\]

into

\[
   E_0[\mathcal{L}_t] = \mathcal{Y}E_0 \left[ Y_{\bar{T}}^T \int_\bar{T}^T e^{-a(s-t) - D_\kappa(s-t)}ds \right]
\]

and apply the Law of Iterated Expectations (conditioning on time \( \bar{T} \) information) to find

\[
   E_0[\mathcal{L}_t] = \mathcal{Y}E_0 \left[ Y_{\bar{T}}^T \int_\bar{T}^T e^{-a(s-t) - D_\kappa(s-t)}(r_\tau e^{-\kappa(t-\bar{T})} + \tilde{\tau}(1 - e^{-\kappa(t-\bar{T})}) - \frac{1}{2}\sigma^2[B_\kappa(s-t)B_\kappa(t-\bar{T})]ds \right].
\]

Next, we substitute (A.14) and (A.16) with \( t = \bar{T} \) in and again use basic properties of stochastic integrals. Tendious computations yield (2.16) with

\[
\begin{align*}
   f_2(t, s) &= r_0e^{-\kappa t} + \tilde{\tau}(1 - e^{-\kappa t}) - \frac{1}{2}\sigma^2\int_0^T e^{-\kappa(t-\bar{T})}B_\kappa(t-u)du - \sigma_\tau\rho_\tau e^{-\kappa(t-\bar{T})}\int_0^T \sigma_Y(u)e^{-\kappa(t-u)}du.
\end{align*}
\]

B Proofs for fully flexible housing decisions

B.1 The HJB equation

Define the scaled controls \( \alpha_S = \pi_S\sigma_Sx, \alpha_B = \pi_B\sigma_Bx, \) and \( \alpha_t = \varphi_t\sigma_Hh. \) Let \( Z = (r, Y, H) \) be the vector of state variables with drift \( \mu_Z = (\kappa[\tilde{\tau} - r], y\mu_Y(r, t), h(r + \lambda_H\sigma_H - r^{imp})) \). Define the vectors \( \lambda = (\lambda_B, \lambda_S, \lambda_H)^T \) and \( \alpha = (\alpha_B, \alpha_S, \alpha_t)^T \), the matrix \( \Sigma \) as in (A.1), and

\[
   \Sigma_Z = \begin{pmatrix}
   -\sigma_r & 0 & 0 \\
   0 & y\sigma_Y(t) & 0 \\
   0 & 0 & h\sigma_H
\end{pmatrix} = \begin{pmatrix}
   -\sigma_r & 0 & 0 \\
   0 & y\sigma_Y(t)\rho_Y & y\sigma_Y(t)\rho_Y \\
   0 & h\sigma_H\rho_H & h\sigma_H\rho_H
\end{pmatrix}.
\]

\( \Sigma \) contains the correlations between the assets \( P = (B, S, H)^T, \) and \( \Sigma_Z \) contains the volatilities and correlations of the state variables \( Z = (r, Y, H). \) Now the dynamics of the state variables and the wealth dynamics from (2.17) can be written compactly as

\[
\begin{align*}
   dZ_t &= \mu_Z(Z_t)dt + \Sigma_Z(Z_t)dW_t, \quad (B.1) \\
   dX_t &= (r_1X_t + \alpha_t^T\lambda_t - \varphi_C\nu H_t - c_t + Y_t)dt + \alpha_t^T\Sigma dW_t, \quad (B.2)
\end{align*}
\]
where $W = (W_r, W_s, W_H)^T$.

The Hamilton-Jacobi-Bellman equation (HJB) associated with the problem can be written as

$$\delta J = \mathcal{L}_1 J + \mathcal{L}_2 J + \mathcal{L}_3 J,$$

where

$$\mathcal{L}_1 J = \max_{c, \varphi_C} \left\{ \frac{1}{1-\gamma} \left( e^{\gamma \varphi_C} \right)^{1-\gamma} - J_x (c + h \varphi_C) \right\},$$

$$\mathcal{L}_2 J = \max_{\alpha} \left\{ J_x \alpha^\top \lambda + \frac{1}{2} J_{xx} \alpha^\top \Sigma \Sigma^\top \alpha + \alpha^\top \Sigma \Sigma^\top J_{xz} \right\},$$

$$\mathcal{L}_3 J = \frac{\partial J}{\partial t} + J_x (r x + y) + J_x^\top \mu_z + \frac{1}{2} \text{tr} (J_{zz} \Sigma \Sigma^\top),$$

where $y$ is replaced by $Y_{\tilde{T}}$ for $t \in [\tilde{T}, T]$.

**B.2 Computation of $\mathcal{L}_1 J$.**

The first-order conditions for $c$ and $\varphi_C$ imply that

$$\beta e^{\beta(1-\gamma)-1} \varphi_C^{(1-\beta)(1-\gamma)} = J_x, \quad (1-\beta) e^{\beta(1-\gamma)-1} \varphi_C^{(1-\beta)(1-\gamma)-1} = \nu h J_x.$$  \hspace{1cm} (B.4)

In particular, $c = \frac{\beta}{1-\beta} \nu h \varphi_C$, so that the relation between optimal perishable consumption and optimal housing consumption is proportional to the relative price of the two goods with a proportionality factor determined by the utility weights of the two goods. Solving the two equations, we find

$$c = \eta \frac{\beta \nu}{1-\beta} h^k J_x^{-1/\gamma}, \quad \varphi_C = \eta h^{k-1} J_x^{-1/\gamma},$$  \hspace{1cm} (B.5)

and

$$\mathcal{L}_1 J = \frac{\gamma}{1-\gamma} \frac{\nu \eta}{1-\beta} h^k J_x^{-1/\gamma}.$$  \hspace{1cm} (B.6)

**B.3 Computation of $\mathcal{L}_2 J$.**

The first-order condition reads

$$J_x \lambda + J_{xx} \Sigma \Sigma^\top \alpha + \Sigma \Sigma^\top J_{xz} = 0$$

or

$$\alpha = -\frac{J_x}{J_{xx}} (\Sigma \Sigma^\top)^{-1} \lambda - \frac{1}{J_{xx}} (\Sigma \Sigma^\top)^{-1} \Sigma \Sigma^\top J_{xz} = -\frac{J_x}{J_{xx}} (\Sigma \Sigma^\top)^{-1} \lambda - \frac{1}{J_{xx}} (\Sigma \Sigma^\top)^{-1} J_{xz}.$$  \hspace{1cm} (B.7)

The dynamics of tangible wealth is then

$$dX_t = \left( r_t X_t - \frac{J_x}{J_{xx}} \hat{\lambda}^\top \hat{\lambda} - \hat{\lambda}^\top \Sigma \Sigma^\top \frac{J_x}{J_{xx}} - [c_t + \nu H_t \varphi_C t] + Y_t \right) dt$$

$$- \frac{J_x}{J_{xx}} \hat{\lambda}^\top dW_t = \frac{J_x}{J_{xx}} \Sigma^\top dW_t,$$

(B.8)
where \( \tilde{\lambda} = \Sigma^{-1} \lambda \). By Itô’s Lemma and (A.12),
\[
dF(t, r_t) = (-1 + F(t, r_t) [r_t - \mu_Y(r_t, t) + \lambda_Y \sigma_Y(t)] - F_t(r_t, t) [\sigma_r \lambda_B - \rho_Y B \sigma_r \sigma_Y(t)]) \, dt
\]
\[
- F_t(r_t, t) \sigma_r \, dW_t.
\]
The dynamics of total wealth, \( W_t = X_t + Y_t F(t, r_t) \), now becomes
\[
dW_t = \left( r_t W_t - J_x \tilde{x} \tilde{\lambda} - J_{xz} \Sigma^T \Sigma Z \Sigma^{-1} \lambda - \frac{1}{2} J_{xz} \Sigma Z \Sigma^T Z J_{xz} \right) \, dt
\]
\[
- \frac{J_x J_{xz}}{J_{xz}} \tilde{x} \tilde{\lambda} \, dW_t - \frac{J_{xz} \Sigma Z}{J_{xz}} \, dW_t - Y_t F_t(r_t, t) \sigma_r \, dW_t + Y_t F_t \sigma_Y(t) \tilde{\rho}_Y \, dW_t.
\]
(B.9)

Substituting the optimal \( \alpha \) back into \( \mathcal{L}_2 J \) leads to
\[
\mathcal{L}_2 J = -\frac{J_x^2}{J_{xz}} \tilde{x} \tilde{\lambda} - \frac{J_x J_{xz} \Sigma^T \Sigma Z}{J_{xz}} \Sigma^{-1} \lambda - \frac{1}{2} \frac{J_{xz} \Sigma Z \Sigma^T Z J_{xz}}{J_{xz}},
\]
(B.10)

where \( \tilde{x} \tilde{\lambda} = \lambda^T \Sigma^{-T} \lambda \). Next, we compute the relevant matrix products. Firstly, since
\[
\begin{pmatrix}
1 & a & b \\
a & 1 & c \\
b & c & 1
\end{pmatrix}
\]

is equal except for the second row. We use the notation
\[
\rho_{SB} = \rho_{xy} - \rho_{xz} \rho_{yz}.
\]

Secondly, disregarding the volatility matrix, \( \Sigma \) and \( \Sigma Z \) are equal except for the second row. We are thus interested in
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

and for any three variables \( x, y, z \) we use the notation
\[
\rho_{xy,z} = \rho_{xy} - \rho_{xz} \rho_{yz}.
\]

Therefore,
\[
\Sigma Z \Sigma^{-1} = \begin{pmatrix}
-\sigma_r & 0 & 0 \\
y \sigma_Y(t) & 0 & 0 \\
0 & h \sigma_H & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
-\sigma_r & 0 & 0 \\
y \sigma_Y(t) & 0 & 0 \\
0 & h \sigma_H & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
-\sigma_r & 0 & 0 \\
y \sigma_Y(t) & 0 & 0 \\
0 & h \sigma_H & 0
\end{pmatrix}
\]
where ζ_B, ζ_S, ζ_I were defined in (A.3)-(A.5).

Thirdly, simple multiplications lead to

\[
\Sigma_Z \Sigma_Z^\top = \begin{pmatrix}
-\sigma_r & 0 & 0 \\
0 & y\sigma_Y(t) & 0 \\
0 & 0 & h\sigma_H
\end{pmatrix} \begin{pmatrix}
1 & \rho_{YB} & \rho_{HB} \\
\rho_{YB} & 1 & \rho_{YH} \\
\rho_{HB} & \rho_{YH} & 1
\end{pmatrix} \begin{pmatrix}
-\sigma_r & 0 & 0 \\
0 & y\sigma_Y(t) & 0 \\
0 & 0 & h\sigma_H
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-\sigma_r^2 & -\rho_{YB}\sigma_r y\sigma_Y(t) & -\rho_{HB}\sigma_r h\sigma_H \\
-\rho_{YB}\sigma_r y\sigma_Y(t) & y^2\sigma_Y(t)^2 & \rho_{YH} y\sigma_Y(t) h\sigma_H \\
-\rho_{HB}\sigma_r h\sigma_H & \rho_{YH} y\sigma_Y(t) h\sigma_H & h^2\sigma_H^2
\end{pmatrix}.
\]

Substitution of these matrix products into (B.7) gives

\[
\alpha_B = \frac{J_x}{J_{xx}}\xi_B - \frac{y J_{xy}}{J_{xx}} y\sigma_Y(t) \xi_B + \frac{J_{xx}}{J_{xx}} \sigma_r,
\]

\[
\alpha_S = -\frac{J_x}{J_{xx}}\xi_S - \frac{y J_{xy}}{J_{xx}} y\sigma_Y(t) \xi_S,
\]

\[
\alpha_I = -\frac{J_x}{J_{xx}}\xi_I - \frac{y J_{xy}}{J_{xx}} y\sigma_Y(t) \xi_I - \frac{J_{xx}}{J_{xx}} h\sigma_H.
\]

where

\[
\xi_B = \frac{1}{\det} \left( \lambda_B (1 - \rho_{2SH}^2) - \rho_{SB, H} \lambda_S - \rho_{BH, S} \lambda_H \right),
\]

\[
\xi_S = \frac{1}{\det} \left( \lambda_S (1 - \rho_{2BH}^2) - \rho_{SB, H} \lambda_B - \rho_{SH, B} \lambda_H \right),
\]

\[
\xi_I = \frac{1}{\det} \left( \lambda_I^2 (1 - \rho_{2SB}^2) - \rho_{SH, B} \lambda_S - \rho_{BH, S} \lambda_B \right).
\]

Note that

\[
\xi^\top \lambda = \lambda^\top (\Sigma \Sigma^\top)^{-1} \lambda = \lambda_B \xi_B + \lambda_S \xi_S + \lambda_I^2 \xi_I.
\]

### B.4 Simplifications when \( J \) has the form in (3.1)

It turns out to be useful to express the derivatives of \( J \) in terms of \( J \) itself:

\[
J_x = (1 - \gamma) \frac{J}{x + yF}, \\
J_y = (1 - \gamma) \frac{F}{x + yF}, \\
J_h = \gamma \frac{g_h}{g}, \\
J_{xy} = -\gamma (1 - \gamma) \frac{F}{(x + yF)^2}, \\
J_{yy} = \gamma (1 - \gamma) \frac{g_h F}{x + yF}, \\
J_{yy} = \gamma (1 - \gamma) \frac{g_h F}{x + yF}, \\
J_{rr} = (1 - \gamma) \frac{\gamma}{1 - \gamma} \frac{g_{rr}}{g} - \gamma \left( \frac{g_r}{g} \right)^2 + 2 \gamma \frac{g_r}{g} \frac{yF_r}{x + yF} - \gamma \left( \frac{yF_r}{x + yF} \right)^2 + \frac{yF_{rr}}{x + yF},
\]
\[ J_{xx} = \gamma(1 - \gamma)J \left[ \frac{g_r}{g} \frac{1}{x + yF} - \frac{yF_r}{(x + yF)^2} \right], \]
\[ J_{xy} = (1 - \gamma)J \left[ \frac{g_r}{g} \frac{F_r}{x + yF} + \frac{F_r}{x + yF} - \gamma \left( \frac{yF}{(x + yF)^2} \right) \right], \]
\[ J_{xh} = (1 - \gamma)J \left[ \frac{1 - \gamma}{g} \frac{g_r h_r}{g^2} + \frac{yF_r}{x + yF} - \gamma \frac{g_h}{g} \right], \]
\[ \frac{\partial J}{\partial t} = (1 - \gamma)J \left[ \frac{\gamma}{1 - \gamma} \frac{\partial g}{\partial t} \frac{1}{g} + \frac{y}{x + yF} \frac{\partial F}{\partial t} \right]. \]

Note that
\[ \frac{J_x}{J_{xx}} = -\frac{1}{\gamma}(x + yF), \quad \frac{J_{xx}}{J_{xx}} = yF - \frac{g_r}{g}(x + yF), \quad \frac{J_{xy}}{J_{xx}} = F, \quad \frac{J_{xh}}{J_{xx}} = -\frac{g_h}{g}(x + yF). \]

Substituting into (B.11)-(B.13), we get the optimal portfolio weights in (3.5) and (3.6) and the optimal housing investment reflected by (3.7). The dynamics of total wealth in (B.9) simplifies to
\[ \frac{dW_t}{W_t} = \left( \gamma + \frac{1}{\gamma} \lambda^\gamma \lambda - \sigma_r \lambda^B \frac{g_r}{g} + \sigma_H \lambda^H H_t - \frac{\sigma_r \lambda^H H_t}{W_t} \right) dt \]
\[ + \frac{1}{\gamma} \lambda^\gamma dW_t - \frac{g_r}{g} \sigma_r dW_{rt} + H_t \frac{g_h}{g} \sigma_H \sigma^H_{H} dW_t, \]

(B.17)

With the conjectured \( J \), the optimal consumption controls in (B.5) simplify to those stated in Eqs. (3.8) and (3.9) in the theorem. Moreover, since
\[ J^{-1-\gamma} = \frac{(1 - \gamma)J}{x + yF} g^{-1}(x + yF) = (1 - \gamma)Jg^{-1}, \]
we get
\[ \mathcal{L}_1J = Jg^{-1} \frac{\gamma \eta^\mu}{1 - \beta h^k}. \]

Substituting the relevant derivatives of \( J \) into (B.10) and simplifying, we obtain
\[ \mathcal{L}_2J = (1 - \gamma)J \left\{ \frac{\lambda^\gamma \lambda}{2 \gamma} + \sigma_r \lambda^B \left( \frac{yF_r}{x + yF} - \frac{g_r}{g} \right) - \sigma_Y(t) \lambda_Y \frac{yF_r}{x + yF} + \sigma_H \lambda^H H_t \frac{g_h}{g} \right. \]
\[ + \frac{\gamma}{2} \sigma^2 \left( \frac{g_r}{g} - \frac{yF_r}{x + yF} \right)^2 + \frac{1}{2} \sigma_Y^2 \left( \frac{y^2 F^2}{(x + yF)^2} \right) + \frac{\gamma}{2} \sigma^2 h^2 \left( \frac{g_h}{g} \right)^2 \]
\[ - \gamma \rho_Y \sigma_r \sigma_Y(t) \frac{yF_r}{x + yF} + \frac{g_r}{g} + \gamma \rho_H \sigma_r \sigma_H \frac{g_h}{g} \left( \frac{yF_r}{x + yF} - \frac{g_r}{g} \right) \]
\[ - \gamma \rho_Y \sigma_r \sigma_Y(t) \frac{yF_r}{x + yF} + \frac{g_h}{g} \} \right. \]

Substituting the relevant derivatives into \( \mathcal{L}_3J \) yields a long expression where a lot of terms are of the form \( (1 - \gamma)Jg/(x + yF) \) multiplied by one of the terms in the PDE (A.12) for \( F \). Due to that
all these terms can be reduced to \((1 - \gamma)J[\sigma_Y(t)\lambda_Y F - \sigma_r \lambda_B F_r]y/(x + yF)\). In total, we get

\[
\mathcal{L}_3 J = (1 - \gamma)J \left\{ \sigma_Y(t)\lambda_Y \frac{yF}{x + yF} - \sigma_r \lambda_B \frac{yF_r}{x + yF} + \frac{\gamma}{1 - \gamma} \frac{\partial g}{\partial t} + r + \frac{1}{1 - \gamma} \kappa \frac{[\tilde{r} - \tau] g_r}{g} \right. \\
+ \frac{\gamma}{1 - \gamma} \left( r + \chi_H \sigma_H - \nu \right) h \frac{g_h}{g} + \frac{\gamma}{2} \sigma_r^2 \left[ \frac{1}{1 - \gamma} \frac{g_{rr}}{g} - \left( \frac{g_r}{g} - \frac{g_{F_r}}{x + yF} \right)^2 \right] \\
- \frac{\gamma}{2} \sigma_Y(t)^2 \frac{y^2F^2}{(x + yF)^2} + \frac{\gamma}{2} \sigma_H^2 h^2 \left[ \frac{1}{1 - \gamma} \frac{g_{hh}}{g} - \left( \frac{g_h}{g} \right)^2 \right] \\
- \gamma \rho_Y \sigma_r \sigma_Y(t) \frac{yF}{x + yF} \left( \frac{g_r}{g} - \frac{g_{F_r}}{x + yF} \right) + \gamma \rho_H \sigma_Y(t) \sigma_H h \frac{g_h}{x + yF} \\
- \left. \gamma \rho_B \sigma_r \sigma_H \left[ \frac{1}{1 - \gamma} \frac{g_{hh}}{g} + \frac{g_h}{g} \left( \frac{yF_r}{x + yF} - \frac{g_r}{g} \right) \right] \right\}.
\]

Summing up, we get

\[
\mathcal{L}_2 J + \mathcal{L}_3 J = \gamma J \frac{1}{g} \left\{ \frac{1}{2} \sigma_r^2 g_{rr} + \frac{1}{2} \sigma_H^2 h^2 g_{hh} - \rho_B \sigma_r \sigma_H h g_{rh} + \left( \kappa [\tilde{r} - \tau] + \frac{\gamma - 1}{\gamma} \sigma_r \lambda_B \right) g_r \\
+ \left( r + \frac{1}{\gamma} \chi_H \sigma_H - \nu \right) h g_h + \frac{\partial g}{\partial t} - \frac{\gamma - 1}{\gamma} \left( r + \frac{\lambda^\gamma \lambda}{2\gamma} \right) g \right\}.
\]

The full HJB-equation now reduces to

\[
0 = \frac{1}{2} \sigma_r^2 g_{rr} + \frac{1}{2} \sigma_H^2 h^2 g_{hh} - \rho_B \sigma_r \sigma_H h g_{rh} + \left( \kappa [\tilde{r} - \tau] + \frac{\gamma - 1}{\gamma} \sigma_r \lambda_B \right) g_r \\
+ \left( r + \frac{1}{\gamma} \chi_H \sigma_H - \nu \right) h g_h + \frac{\partial g}{\partial t} + \frac{\nu}{1 - \beta} h k - \left( \frac{\delta}{\gamma} + \frac{\gamma - 1}{\gamma} r + \frac{\gamma - 1}{2\gamma^2} \lambda^\gamma \lambda \right) g
\]

with the terminal condition \(g(T, r, h) = \varepsilon^{1/\gamma}\). Conjecturing a solution of the form

\[
g(t, r, h) = \varepsilon^{1/\gamma} e^{-d_0(T-t)-d_1(T-t)r} + \frac{\eta \nu}{1 - \beta} \int_t^T e^{-d_1(u-t) - d_1(u-t)r} \, du,
\]

we find that \(d_0\) and \(\tilde{d}_0\) must satisfy the ODEs

\[
d_0'(\tau) + \kappa d_0(\tau) = \frac{\gamma - 1}{\gamma}, \\
d_0'(\tau) = -\frac{1}{2} \sigma_r^2 d_0(\tau)^2 + \left( \kappa \tilde{r} + \frac{\gamma - 1}{\gamma} \sigma_r \lambda_B \right) d_0(\tau) + \frac{\delta}{\gamma} + \frac{\gamma - 1}{2\gamma^2} \lambda^\gamma \lambda
\]

with \(d_0(0) = \tilde{d}_0(0) = 0\), and \(d_1\) and \(\tilde{d}_1\) must satisfy the ODEs

\[
d_1'(\tau) + \kappa \tilde{d}_1(\tau) = \frac{\gamma - 1}{\gamma} - k = \beta \frac{1}{\gamma} - 1, \\
d_1'(\tau) = -\frac{1}{2} \sigma_r^2 \tilde{d}_1(\tau)^2 + \left( \kappa \tilde{r} + \frac{\gamma - 1}{\gamma} \sigma_r \lambda_B - k \sigma_r \sigma_H \rho_B \right) \tilde{d}_1(\tau) \\
+ \frac{\delta}{\gamma} + \frac{\gamma - 1}{2\gamma^2} \lambda^\gamma \lambda - \frac{1}{2} k(k-1) \sigma_H^2 k - \left[ \frac{1}{\gamma} \sigma_H \chi_H - \nu \right]
\]

with \(d_1(0) = \tilde{d}_1(0) = 0\). The solutions for \(d_0\) and \(\tilde{d}_1\) are

\[
d_0(\tau) = \frac{\gamma - 1}{\gamma} \kappa \left( 1 - e^{-\kappa \tau} \right) = \frac{\gamma - 1}{\gamma} B(\kappa)(\tau), \\
\tilde{d}_1(\tau) = \beta \frac{1}{\gamma} - 1 \kappa \left( 1 - e^{-\kappa \tau} \right) = \beta \frac{1}{\gamma} - 1 B(\kappa)(\tau).
\]
Straightforward integration yields
\[ d_0(\tau) = \left( \frac{\delta}{\gamma} + \frac{\gamma - 1}{2\gamma^2} \right) \tau - \frac{1 - \gamma}{2} \sigma_r^2 \left( \frac{\gamma - 1}{\gamma} \right)^2 \int_0^\tau B_\kappa(u)^2 du + \left( \kappa \bar{r} + \frac{\gamma - 1}{\gamma} \sigma_r \lambda_B \right) \frac{\gamma - 1}{\gamma} \int_0^\tau B_\kappa(u) du, \]
which—exploiting the integration results in Appendix D—gives \( d_0(\tau) = D_\gamma(\tau) \) defined in (3.3). The expression (3.4) for \( d_1(\tau) \) follows analogously.

### B.5 Expected consumption and wealth

We first consider the simpler case with \( \varepsilon = 0 \) and afterwards generalize to \( \varepsilon > 0 \). We define \( \hat{\beta} = \beta(\gamma - 1)/\gamma \).

(a) \( \varepsilon = 0 \): The total wealth dynamics is given in (3.13), optimal perishable consumption is \( c_t = \beta W_t/G(t,r_t) \), and optimal expenditure on housing consumption is \( \varphi_{Ct} \nu H_t = (1 - \beta)W_t/G(t,r_t) \). We will use Itô’s Lemma to find the dynamics of \( W_t/G(t,r_t) \) and then compute expectations.

First we establish the dynamics of \( W_t/G(t,r_t) \), where \( M_t^\mu = \exp\{-d_1(u-t) - \beta B_\kappa(u-t)r_t\} \).

Since
\[ dM_t^\mu = M_t^\mu \left\{ \left( \hat{\beta}r_t + \left( \frac{\gamma - 1}{\gamma} \sigma_r \lambda_B - k \sigma_r \sigma_H \rho_H \right) \right) \frac{\gamma - 1}{\gamma} \sigma_H \lambda_B \right\} dt + \hat{\beta}B_\kappa(u-t) \sigma_r dW_t, \]
we get
\[ \frac{dG(t,r_t)}{G(t,r_t)} = \left[ -\frac{1}{G(t,r_t)} + \left( \hat{\beta}r_t + \frac{\delta}{\gamma} + \frac{\gamma - 1}{2\gamma^2} \lambda^\top \lambda - \frac{1}{2} k(k-1) \sigma_H^2 - k \left( \frac{1}{\gamma} \sigma_H \lambda_H - \nu \right) \right) \right] dt + \hat{\beta}B_\kappa(u-t) \sigma_r dW_t. \]

Now Itô’s Lemma implies
\[ \frac{d(W_t/G(t,r_t))}{W_t/G(t,r_t)} = \frac{dW_t}{W_t} - \frac{dG(t,r_t)}{G(t,r_t)} + \left( \frac{dG(t,r_t)}{G(t,r_t)} \right)^2 - \frac{dW_t}{W_t} \frac{dG(t,r_t)}{G(t,r_t)} = \left[ \left( 1 - \hat{\beta} \right) r_t + \frac{1 + \gamma}{2\gamma^2} \lambda^\top \lambda + k \sigma_H \lambda_H - \frac{\delta}{\gamma} + \frac{1}{2} k(k-1) \sigma_H^2 + k \left( \frac{1}{\gamma} \sigma_H \lambda_H - \nu \right) \right] dt \]
\[ + \frac{1}{\gamma} \lambda^\top dW_t + k \sigma_H \rho_H dW_t. \]

We can write \( \frac{W_t}{G(t,r_t)} = \frac{W_t}{G(0,r_0)} e^{\lambda_\tau} \) and use \( \lambda \nu \) to conclude that the random variable \( \hat{z} \) is normally distributed. Taking expectations, we get
\[ E_0 \left[ \frac{W_t}{G(t,r_t)} \right] = \frac{W_0}{G(0,r_0)} e^{A(t)}, \]
where
\[ A(t) = \left[ \left( 1 - \hat{\beta} \right) r_0 + \frac{1 + \gamma}{2\gamma^2} \lambda^\top \lambda + k \sigma_H \lambda_H - \frac{\delta}{\gamma} + \frac{1}{2} k(k-1) \sigma_H^2 + k \left( \frac{1}{\gamma} \sigma_H \lambda_H - \nu \right) \right] t \]
\[ + \left( 1 - \hat{\beta} \right) \left[ \bar{r} - r_0 + \frac{\sigma_r^2 (1 - \beta)}{2k^2} - \frac{\sigma_r}{\kappa} \left( \frac{1}{\gamma} \lambda_B + k \sigma_H \rho_H \right) \right] (t - B_\kappa(t)) - \left( 1 - \hat{\beta} \right)^2 \frac{\sigma_r^2}{4k^2} B_\kappa(t)^2. \]

\(^{19}\) If \( dM_t^r = \mu_t^r du + \sigma_t^r dW_t \) and \( G_t = \int_0^t M_t^r du \), then \( dG_t = \left( -M_t^r + \int_0^\tau \mu_t^r du \right) dt + \left( \int_0^\tau \sigma_t^r du \right) dW_t. \)
The expected consumption rate at time $t$ is then $\text{Eq}[c_t] = \beta \frac{W_0}{G(0,r_0)} e^{A(t)}$ and the expected spending on housing consumption is $\text{Eq} [\phi_C(\nu H_t)] = (1 - \beta) \frac{W_0}{G(0,r_0)} e^{A(t)}$.

Apparently, it is impossible to find a precise explicit expression for the expected total wealth, $\text{Eq}[W_t]$, when total wealth dynamics is given in (3.13), but we can derive an approximate expression as follows. Experiments show that $D(t,r_t)$ and $G(t,r_t)$ are very little sensitive to $r_t$ so we replace them by $\bar{D}(t) = D(t, \bar{r})$ and $\bar{G}(t) = G(t, \bar{r})$, respectively. Again applying (A.9), we find that future (approximate) total wealth is lognormally distributed with

$$
\text{E}_0[W_t] \approx W_0 \exp \left\{ \left[ r_0 + \frac{1}{\gamma} \bar{\lambda}^+ \bar{\lambda} + k \sigma_H \lambda'_H \right] t + \left( \tilde{r} - r_0 - \frac{\sigma_r}{\kappa} \left[ \frac{\lambda_B}{\bar{\lambda}} + k \sigma_H \mu_H - \frac{\sigma_r}{2\kappa} \right] \right) (t - B_k(t)) \right.

- \frac{\sigma_r^2}{4\kappa} B_k(t)^2 + \sigma_r \lambda_B \beta \int_0^t \bar{D}(u) du - \int_0^t \bar{G}(u)^{-1} du - \beta \sigma_r^2 \int_0^t \bar{D}(u) B_k(t - u) du \right\},
$$

where the integrals have to evaluated numerically.

(b) $\varepsilon > 0$: The optimal perishable consumption at time $t$ is

$$
c_t = \eta \frac{\beta \nu}{1 - \beta} \frac{W_t H_t^k}{g(t, r_t, H_t)} = \frac{\beta \nu}{1 - \beta} \frac{W_t}{G(t, r_t, H_t)},
$$

where $g$ is defined in (3.2) and

$$
G(t, r, h) = e^{\frac{\varepsilon}{2} h - \frac{k}{2} \frac{\gamma - 1}{\gamma} h - \frac{\delta}{2}} \mathcal{K}_0(t) + \frac{\eta \nu}{1 - \beta} \int_0^t e^{-d_1(u-t) - \beta B_k(u-t) r} du.
$$

Tedious computations along the same lines as above yield that

$$
\frac{d}{dt} \left( \frac{W_t H_t^k}{g(t, r_t, H_t)} \right) = \left[ (1 - \beta) r_t + \frac{1}{2 \gamma^2} \bar{\lambda}^+ \bar{\lambda} + k \sigma_H \lambda'_H - \frac{\delta}{\gamma} + \frac{1}{2} k (k - 1) \sigma_H^2 + k \left( \frac{1}{\gamma} \sigma_H \lambda'_H - \nu \right) \right] dt

+ \frac{1}{\gamma} \lambda^+ dW_t + k \sigma_H \mu_H dW_t,
$$

analogously to (B.19). Therefore, we can conclude that

$$
\text{E}_0 \left[ \frac{W_t H_t^k}{g(t, r_t, H_t)} \right] = \frac{W_0 H_0^k}{g(0, r_0, H_0)} e^{A(t)},
$$

where $A(t)$ is as defined above, from which the time 0 expectations of time $t$ consumption follows easily.

Expected wealth itself is approximated similarly as above. The total wealth dynamics is now

$$
\frac{dW_t}{W_t} = \left( r_t + \frac{1}{\gamma} \bar{\lambda}^+ \bar{\lambda} + \sigma_r \lambda_B D(t, r_t, H_t) + k \sigma_H \lambda_H' \frac{\eta \nu}{1 - \beta} H(t, r_t, H_t) + \frac{\eta \nu}{1 - \beta} G(t, r_t, H_t)^{-1} \right) dt

+ \frac{1}{\gamma} \lambda^+ dW_t + \frac{\gamma - 1}{\gamma} \sigma_r D(t, r_t, H_t) dW_{rt} + k \sigma_H \frac{\eta \nu}{1 - \beta} H(t, r_t, H_t) \mu_H dW_t,
$$

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where

\[
D(t,r,h) = G(t,r,h)^{-1} \left( e^{\frac{1}{2} h^{-1} \mathcal{B}_\kappa(T-t)e^{-\mathcal{D}_\kappa(T-t)} - \mathcal{L}_\kappa(T-t)} \right) + \frac{\eta \nu \beta}{1 - \beta} \int_t^T \mathcal{B}_\kappa(u-t)e^{-d_1(u-t) - \beta \mathcal{B}_\kappa(u-t)r} du
\]

\[
\mathcal{H}(t,r,h) = G(t,r,h)^{-1} \int_t^T e^{-d_1(u-t) - \beta \mathcal{B}_\kappa(u-t)r} du.
\]

Numerical experiments show that \( G, D, \) and \( \mathcal{H} \) are only little sensitive to \( r \) and \( h \), so we use the approximations

\[
G(t,r_t,H_t) \approx \tilde{G}(t) \equiv G(t,r_t,E[H_t]),
\]

\[
D(t,r_t,H_t) \approx \tilde{D}(t) \equiv D(t,r_t,E[H_t]),
\]

\[
\mathcal{H}(t,r_t,H_t) \approx \tilde{\mathcal{H}}(t) \equiv \mathcal{H}(t,r_t,E[H_t])
\]

in the wealth dynamics above. Applying (A.9), we find that future (approximate) total wealth is lognormally distributed with

\[
E_0[W_t] \approx W_0 \exp \left\{ \left( r_0 + \frac{1}{\gamma} \lambda^\lambda \right) t + \left( \bar{r} - r_0 - \frac{\sigma^r}{\kappa} \left( \frac{\lambda^B}{\gamma} - \frac{\sigma^r}{2\kappa} \right) \right) (t - B\kappa(t)) - \frac{\sigma^2}{4\kappa} B\kappa(t)^2 \right\}
\]

\[
+ \frac{\gamma - 1}{\gamma} \sigma^r \lambda^B \int_0^t \tilde{\mathcal{D}}(u) du + k \sigma_H \lambda_H^\gamma \frac{\eta \nu}{1 - \beta} \int_0^t \tilde{\mathcal{H}}(u) du - \frac{\eta \nu}{1 - \beta} \int_0^t \tilde{\mathcal{G}}(u)^{-1} du
\]

\[- \frac{\gamma - 1}{\gamma} \sigma^r \int_0^t \tilde{\mathcal{D}}(u) B\kappa(t-u) du - k \rho_H \sigma_r \sigma_H \frac{\eta \nu}{1 - \beta} \int_0^t \tilde{\mathcal{H}}(u) B\kappa(t-u) du \}
\]

(B.21)

where the integrals have to be evaluated numerically.

C Proof of Theorem 5.1 (constant housing consumption)

The HJB equation is again of the form (B.3) with \( \mathcal{L}_2 J \) still given by (B.10), while

\[
\mathcal{L}_1 J = \max_c \left\{ \frac{1}{1 - \gamma} \left( e^{\beta \varphi C - \beta} \right)^{1-\gamma} - c J_x \right\},
\]

\[
\mathcal{L}_3 J = \frac{\partial J}{\partial t} + J_x (r x + y) + J^2_z \mu_z + \frac{1}{2} \text{tr} (J z_z \Sigma z_z \Sigma^T z) - x_z h \nu \varphi C.
\]

With a conjecture of the form (5.3) for the value function, the derivatives of \( J \) can be written in terms of \( J \) as follows:

\[
J_x = \frac{(1 - \gamma)}{x + y F - \nu h F^2},
\]

\[
J_y = \frac{(1 - \gamma)}{x + y F - \nu h F^2},
\]

\[
J_h = \frac{(1 - \gamma)}{x + y F - \nu h F^2},
\]

\[
J_{xx} = -\frac{\gamma(1 - \gamma) J}{(x + y F - \nu h F)^2},
\]

\[
J_{yy} = -\frac{\gamma(1 - \gamma) J}{(x + y F - \nu h F)^2},
\]

\[
J_{hh} = -\frac{\gamma(1 - \gamma) J}{(x + y F - \nu h F)^2},
\]

\[
\left( \frac{\nu h}{x + y F - \nu h F} \right)^2,
\]

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\[ J_{xy} = \tilde{\gamma}(1 - \tilde{\gamma})J \frac{F}{(x + yF - \nu h F)^2}, \quad J_{xh} = \tilde{\gamma}(1 - \tilde{\gamma})J \frac{\nu F}{(x + yF - \nu h F)^2}, \]
\[ J_{hy} = \tilde{\gamma}(1 - \tilde{\gamma})J \frac{\nu F}{(x + yF - \nu h F)^2}, \quad J_r = (1 - \tilde{\gamma})J \left[ \frac{\tilde{\gamma} g}{1 - \tilde{\gamma} g} + \frac{yF_r}{x + yF - \nu h F} \right], \]
\[ \frac{\partial J}{\partial t} = (1 - \tilde{\gamma})J \left[ \frac{\tilde{\gamma}}{1 - \tilde{\gamma}} \frac{\partial g}{\partial t} + \frac{1}{x + yF - \nu h F} \left( \frac{\partial F}{\partial t} - \nu h F \right) \right], \]
\[ J_{rr} = (1 - \tilde{\gamma})J \left[ \frac{\tilde{\gamma} g}{1 - \tilde{\gamma} g} \left( \frac{g_r}{g} \right)^2 + \frac{\tilde{\gamma} \gamma g}{g} \frac{yF_r}{x + yF - \nu h F} \right], \]
\[ J_{xx} = \tilde{\gamma}(1 - \tilde{\gamma})J \left[ \frac{g_r}{g} \frac{1}{x + yF - \nu h F} - \frac{yF_r}{(x + yF - \nu h F)^2} \right], \]
\[ J_{yy} = (1 - \tilde{\gamma})J \left[ \frac{\gamma g_r}{g} \frac{F}{x + yF - \nu h F} - \tilde{\gamma} \frac{F y F_r}{(x + yF - \nu h F)^2} + \frac{F_r}{x + yF - \nu h F} \right], \]
\[ J_{hh} = (1 - \tilde{\gamma})J \left[ -\tilde{\gamma} \frac{g_r}{g} \frac{\nu F}{x + yF - \nu h F} + \tilde{\gamma} \frac{\nu F y F_r}{(x + yF - \nu h F)^2} \right]. \]

Note that \[ \frac{J_x}{\partial x} = \frac{-1}{\tilde{\gamma}}(x + yF - \nu h F), \quad \frac{J_{xx}}{\partial x} = \frac{yF_r}{g} \frac{1}{(x + yF - \nu h F)}, \quad \frac{J_{xy}}{\partial x} = \frac{F}{yF_r}, \quad \frac{J_{xh}}{\partial x} = \frac{\nu h F}{yF_r}. \]

The first-order conditions for \( \alpha_B, \alpha_S, \) and \( \alpha_t \) are still given by (B.11)-(B.13). Substituting in the relevant derivatives from above, we obtain (5.6)-(5.8).

We have
\[ L_1 J = \max_c \left\{ \frac{1}{1 - \gamma} e^{\beta(1 - \gamma) \phi C (1 - \gamma)} - c J_x \right\} \]
giving a first-order condition implying that
\[ c = \beta^{1/\tilde{\gamma}} \phi C^{1-\gamma/\tilde{\gamma}} J_x^{-1/\tilde{\gamma}} \]
(C.1)
from which we obtain
\[ L_1 J = \frac{\tilde{\gamma}}{1 - \tilde{\gamma}} \beta^{1/\tilde{\gamma}} \phi C^{1-\gamma/\tilde{\gamma}} J_x^{-1/\tilde{\gamma}}. \]

With the conjecture for \( J \), we get the expression (5.9) for \( c \) and
\[ L_1 J = \tilde{\gamma} g^{-1} \beta^{1/\tilde{\gamma}} \phi C^{1-\gamma/\tilde{\gamma}}. \]

Substituting the relevant derivatives of \( J \) into (B.10) and simplifying, we obtain
\[ L_2 J = (1 - \tilde{\gamma})J \left\{ \tilde{\gamma} \frac{T_{\tilde{\gamma}}}{2 \tilde{\gamma}} + \sigma_r \lambda_B \left( \frac{yF_r}{x + yF - \nu h F} - \frac{g_r}{g} \right) - \sigma_f(t) \lambda_Y \frac{yF}{x + yF - \nu h F} + \sigma_H \lambda_H \frac{\nu h F}{x + yF - \nu h F} \right. \]
\[ \left. + \frac{\tilde{\gamma}}{2 \sigma_r^2} \left( \frac{g_r}{g} - \frac{yF_r}{x + yF - \nu h F} \right)^2 + \frac{\tilde{\gamma}}{2} \sigma_Y(t) \frac{y^2 F^2}{(x + yF - \nu h F)^2} + \frac{\tilde{\gamma}}{2} \sigma_H^2 \left( \frac{\nu h F}{x + yF - \nu h F} \right)^2 \right. \]
\[ - \tilde{\gamma} \rho_{Y B} \sigma_r \sigma_Y(t) \left( \frac{yF_r}{x + yF - \nu h F} - \frac{g_r}{g} \right) \]
\[ \left. - \tilde{\gamma} \rho_{H B} \sigma_r \sigma_H \frac{\nu h F}{x + yF - \nu h F} \left( \frac{g_r}{g} - \frac{yF_r}{x + yF - \nu h F} \right) - \tilde{\gamma} \rho_{Y H} \sigma_H \sigma_Y(t) \frac{yF_v h F}{(x + yF - \nu h F)^2} \right\}. \]
Substituting the relevant derivatives into $L_3J$ and applying the PDE (A.12) for $F$ and the fact that 
\( \dot{F}'(t) = -\varphi_C + \nu \dot{F}(t) \), we obtain

\[
L_3J = (1 - \gamma) \left\{ \sigma_Y(t) \lambda \gamma \left\{ \frac{yF}{x + yF - \nu F} - \sigma_r \lambda_B \frac{yF_r - \nu F}{x + yF - \nu F} - \sigma_H \lambda_H' \frac{\nu F}{x + yF - \nu F} \right\} \right. \\
+ \frac{\gamma}{1 - \gamma} \frac{\partial}{\partial t} \left[ \frac{g_{rr} + \gamma(\gamma - 1) \sigma_r \lambda_B}{g} \right] \left. \left\{ \frac{1}{1 - \gamma} \frac{g_{rr}}{g} - \frac{g_{rr}}{g} \frac{yF_r}{x + yF - \nu F} \right\} \right. \\
- \frac{\gamma}{2} \frac{\gamma}{1 - \gamma} \frac{g_{rr} + \gamma(\gamma - 1) \sigma_r \lambda_B}{g} \left. \left\{ \frac{1}{1 - \gamma} \frac{g_{rr}}{g} - \frac{g_{rr}}{g} \frac{yF_r}{x + yF - \nu F} \right\} \right. \\
+ \frac{\gamma}{2} \frac{\gamma}{1 - \gamma} \frac{g_{rr} + \gamma(\gamma - 1) \sigma_r \lambda_B}{g} \left. \left\{ \frac{1}{1 - \gamma} \frac{g_{rr}}{g} - \frac{g_{rr}}{g} \frac{yF_r}{x + yF - \nu F} \right\} \right. \}
\]

Summing up, we get

\[
L_2J + L_3J = \gamma J \frac{1}{g} \left\{ \frac{1}{2} \sigma_r^2 g_{rr} + \left( \kappa [\dot{r} - r] + \frac{\gamma - 1}{\gamma} \sigma_r \lambda_B \right) g_r + \frac{\partial g}{\partial t} - \frac{\gamma - 1}{\gamma} \left( r + \frac{\lambda^\tau \lambda}{2\gamma} \right) \right\}.
\]

Substituting into $\delta J = L_1J + L_2J + L_3J$, we see that $g$ must satisfy the PDE

\[
0 = \frac{1}{2} \sigma_r^2 g_{rr} + \left( \kappa [\dot{r} - r] + \frac{\gamma - 1}{\gamma} \sigma_r \lambda_B \right) g_r + \frac{\partial g}{\partial t} - \frac{\gamma - 1}{\gamma} \left( r + \frac{\lambda^\tau \lambda}{2\gamma} \right) g + \beta^{\frac{1}{2}} \varphi_C^{1 - \frac{r}{2}} - \frac{r}{2\gamma} \left( \kappa \frac{\lambda^\tau \lambda}{2\gamma} \right) g + \beta^{\frac{1}{2}} \varphi_C^{1 - \frac{r}{2}} \quad (C.2)
\]

with the terminal condition $g(T, r) = 0$ (since $\varepsilon = 0$). Since this resembles the pricing PDE for an asset with a zero terminal payment and a continuous dividend of $\beta^{\frac{1}{2}} \varphi_C^{1 - \frac{r}{2}}$, we try a solution of the form

\[
g(t, r) = \beta^{\frac{1}{2}} \varphi_C(s)^{1 - \frac{r}{2}} \int_0^T e^{-d(s-t) - \bar{d}(s-t)} ds.
\]

We need $d(0) = \bar{d}(0) = 0$ and

\[
d(\tau) + \kappa d(\tau) = 1 - \frac{1}{\gamma}, \\
d'(\tau) = -\frac{1}{2} \sigma_r^2 d(\tau)^2 + \left( \kappa \dot{r} + \frac{\gamma - 1}{\gamma} \sigma_r \lambda_B \right) d(\tau) + \frac{\delta}{\gamma} + \frac{\gamma - 1}{2\gamma^2} \lambda^\tau \lambda.
\]

The solution to the first ODE is $d(\tau) = \frac{2\gamma - 1}{\gamma} B_\kappa(\tau)$ and then $d(\tau)$ follows from the second ODE by integration:

\[
d(\tau) = -\frac{1}{2} \sigma_r^2 \left( \frac{\gamma - 1}{\gamma} \right)^2 \int_0^\tau B_\kappa(u)^2 du + \frac{\gamma - 1}{\gamma} \left( \kappa \dot{r} + \frac{\gamma - 1}{\gamma} \sigma_r \lambda_B \right) \int_0^\tau B_\kappa(u) du + \left( \frac{\delta}{\gamma} + \frac{\gamma - 1}{2\gamma^2} \lambda^\tau \lambda \right) \tau.
\]

Using the integration results of Appendix D we obtain $d(\tau) = D_\gamma(\tau)$ defined through (3.3).

### D Some properties of the $B$-function

Recall the definition

\[
B_m(\tau) = \frac{1 - e^{-\omega \tau}}{m}
\]

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Lemma D.1 (Multiplying Bs) The following equations hold
\[
B_bB_c = \frac{bb_b + cb_c - (b + c)B_{b+c}}{bc} \quad (D.1)
\]
\[
B_{b+c} = \frac{bb_b + cb_c - bcB_bB_c}{b+c}.
\]

**Proof.** Follows from the definition of \(B\). \(\square\)

**Remark.** Equation (D.1) shows that second-order terms of the form \(B_bB_c\) can be transformed into a sum of first-order terms. Note that, in particular, \(B_{2b} = B_b - \frac{b}{2}B^2_b\).

We set \(\tau = T - t\).

Lemma D.2 (Integrating Bs) (i) Assuming \(a \neq 0\), we obtain\(^{20}\)
\[
\int_t^T e^{-a(s-t)}B_b(T-s)\, ds = \begin{cases} 
\frac{B_a(\tau)-B_a(\tau)}{b-a} & \text{if } b \neq a, \\
\frac{B_a(\tau) + (aB_a(\tau) - 1)\tau}{a} & \text{if } b = a,
\end{cases} \quad (D.2)
\]
\[
\int_t^T e^{-a(s-t)}B_b(T-s)B_c(T-s)\, ds = \begin{cases} 
\frac{1}{bc} \left[ \frac{bc(b+c-2a)}{(b-a)(c-a)(b+c-a)} B_a(\tau) - \frac{b}{b-a} B_c(\tau) - \frac{c}{c-a} B_a(\tau) + \frac{b+c}{b+c-a} B_{b+c}(\tau) \right] & \text{if } b \neq a \neq c, \\
\frac{1}{b} \left[ \frac{a}{b-a} B_a(\tau) - \frac{b}{b-a} B_b(\tau) + \frac{a+b}{b} B_{a+b}(\tau) + (aB_a(\tau) - 1)\tau \right] & \text{if } b \neq a = c, \\
\frac{1}{a} \left[ 2B_a(\tau) + (aB_a(\tau) - 1)\tau \right] & \text{if } b = a = c.
\end{cases} \quad (D.3)
\]

(ii) Furthermore,
\[
\int_t^T B_b(T-s)\, ds = \frac{\tau - B_b(\tau)}{b}
\]
\[
\int_t^T B_b(T-s)B_c(T-s)\, ds = \frac{\tau - B_b(\tau) - B_c(\tau) + B_{b+c}(\tau)}{bc} \quad (D.4)
\]

**Proof.** Equation (D.2) follows by simple integration. To show (D.3), one can use (D.1) and then apply (D.2). The proof of (ii) works similarly. \(\square\)

**Remarks.**

a) In the special case \(a \neq b = c\), equation (D.3) simplifies into
\[
\frac{2}{b} \left[ \frac{b}{(b-a)(2b-a)} B_a(\tau) - \frac{1}{b-a} B_b(\tau) + \frac{1}{2b-a} B_{2b}(\tau) \right].
\]
b) In the special case \(b = c\), equation (D.4) simplifies into
\[
\frac{1}{b^2} \left[ \tau - 2B_b(\tau) + B_{2b}(\tau) \right] = \frac{1}{b^2} \left[ \tau - B_b(\tau) - \frac{b}{2} B_b(\tau)^2 \right].
\]
c) We can use (D.1) to rewrite (D.3) as follows:
\[
\int_t^T e^{-a(s-t)}B_b(T-s)B_c(T-s)\, ds = \begin{cases} 
\frac{1}{b+c-a} \left[ \frac{b+c-2a}{b-a} B_a(\tau) - \frac{1}{c-a} B_c(\tau) - B_b(\tau)B_c(\tau) \right] & \text{if } b \neq a \neq c, \\
\frac{1}{b-a} \left[ \frac{b-a}{b} B_a(\tau) - \frac{a}{b} B_b(\tau) - aB_a(\tau)B_b(\tau) + (aB_a(\tau) - 1)\tau \right] & \text{if } b \neq a = c, \\
\frac{1}{a^2} \left[ (2 - aB_a(\tau))B_a(\tau) + 2(aB_a(\tau) - 1)\tau \right] & \text{if } b = a = c.
\end{cases}
\]

\(^{20}\)Note that \(b \neq a \neq c\) does not exclude cases with \(b = c\).
References


Tsai, I.-C., M.-C. Chen, and T.-F. Sing (2007, November). Do REITs behave more like real estate now? Working paper, Southern Taiwan University of Technology, National Sun Yat-sen University of Taiwan and National University of Singapore.


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Figure 1: **Expected consumption over the life-cycle.** The two solid curves show the expected spending in thousands of US dollars on the perishable consumption good (dark curve) and on housing consumption (pale curve) with values to be read off the left-side axis. The dashed line shows the expected number of housing units (“standard square feet”) consumed and is to be read off the right-side axis. The results are generated with the benchmark parameters in Table 1. The current short rate is set to the long-term level, $r = \bar{r}$, the current house price is $h = 250$ USD per housing unit, the current financial wealth is $x = 20,000$ USD, and the current income is $y = 20,000$ USD per year.
Figure 2: Optimal investments and the composition of wealth. The figure shows how the optimal fractions of total wealth invested in bonds, stocks, and houses (physically or financially) vary with the ratio of human wealth $yF(t, r)$ to total wealth $W = x + yF(t, r)$. The benchmark parameters in Table 1 are used. The ratios $F_t/F$ and $g_t/g$ are computed assuming 30 years to retirement and a retirement period of 20 years. The current short rate is set to the long-term level, $r = \bar{r}$. 
Figure 3: Expected wealth over the life-cycle. The figure shows the initial expectations of total wealth, financial wealth, and human wealth over the life-cycle. The results are generated with the benchmark parameters in Table 1. The current short rate is set to the long-term level, $r = \bar{r}$, the current house price is $h = 250$ USD per housing unit, the current financial wealth is $x = 20,000$ USD, and the current income is $y = 20,000$ USD per year.
Figure 4: **Expected investments over the life-cycle.** The figure shows the initial expectations of the investments in bonds, stocks, and housing units over the life-cycle. The results are generated with the benchmark parameters in Table 1. The current short rate is set to the long-term level, \( r = \bar{r} \), the current house price is \( h = 250 \text{ USD per housing unit} \), the current financial wealth is \( x = 20,000 \text{ USD} \), and the current income is \( y = 20,000 \text{ USD per year} \).
Figure 5: **Consumption of and investment in housing over the life-cycle.** The figure shows the initial expectations of the number of housing units consumed and the number of housing units invested in physically or via financial contracts over the life-cycle. The results are generated with the benchmark parameters in Table 1. The current short rate is set to the long-term level, \( r = \bar{r} \), the current house price is \( h = 250 \) USD per housing unit, the current financial wealth is \( x = 20,000 \) USD, and the current income is \( y = 20,000 \) USD per year.
Figure 6: Expected investments over the life-cycle with and without bequest. The figure shows the initial expectations of the investments in bonds, stocks, and housing units over the life-cycle. The gray curves depict results with no utility from terminal wealth, $\varepsilon = 0$, whereas the black curves are for the case with utility from terminal wealth given by $\varepsilon = 10$. For other parameters, the benchmark values in Table 1 are applied. The current short rate is set to the long-term level, $r = \bar{r}$, the current house price is $h = 250$ USD per housing unit, the current financial wealth is $x = 20,000$ USD, and the current income is $y = 20,000$ USD per year.
Figure 7: **Expected investments over the life-cycle for different risk aversions.** The figure shows the initial expectations of the investments in bonds, stocks, and housing units over the life-cycle. The medium-dark gray curves are for the benchmark relative risk aversion of $\gamma = 4$, the black curves are for a lower risk aversion of $\gamma = 2$, whereas the light gray curves are for a higher risk aversion of $\gamma = 6$. For other parameters, the benchmark values in Table 1 are applied. The current short rate is set to the long-term level, $r = \bar{r}$, the current house price is $h = 250$ USD per housing unit, the current financial wealth is $x = 20,000$ USD, and the current income is $y = 20,000$ USD per year.
Figure 8: Consumption of and investment in housing over the life-cycle for different risk aversions. The figure shows the initial expectations of the number of housing units consumed and the number of housing units invested in physically or via financial contracts over the life-cycle. The medium-dark gray curves are for the benchmark relative risk aversion of $\gamma = 4$, the black curves are for a lower risk aversion of $\gamma = 2$, whereas the light gray curves are for a higher risk aversion of $\gamma = 6$. For other parameters, the benchmark values in Table 1 are applied. The current short rate is set to the long-term level, $r = \bar{r}$, the current house price is $h = 250$ USD per housing unit, the current financial wealth is $x = 20,000$ USD, and the current income is $y = 20,000$ USD per year.
Figure 9: Expected investments over the life-cycle for different house price volatilities. The figure shows the initial expectations of the investments in bonds, stocks, and housing units over the life-cycle. The medium-dark gray curves are for the benchmark house price volatility of $\sigma_H = 0.12$, the black curves are for a lower volatility of $\sigma_H = 0.08$, whereas the light gray curves are for a higher volatility of $\sigma_H = 0.16$. For other parameters, the benchmark values in Table 1 are applied. The current short rate is set to the long-term level, $r = \bar{r}$, the current house price is $h = 250$ USD per housing unit, the current financial wealth is $x = 20,000$ USD, and the current income is $y = 20,000$ USD per year.
Figure 10: **Expected consumption over the life-cycle for different house price volatilities.** The dashed and the dashed-dotted curves show the expected spending in thousands of US dollars on the perishable consumption good and on housing consumption with values to be read off the left-side axis. The solid curves show the expected number of housing units (“standard square feet”) consumed and are to be read off the right-side axis. The medium-dark gray curves are for the benchmark house price volatility of $\sigma_H = 0.12$, the black curves are for a lower volatility of $\sigma_H = 0.08$, whereas the light gray curves are for a higher volatility of $\sigma_H = 0.16$. For other parameters, the benchmark values in Table 1 are applied. The current short rate is set to the long-term level, $r = \bar{r}$, the current house price is $h = 250$ USD per housing unit, the current financial wealth is $x = 20,000$ USD, and the current income is $y = 20,000$ USD per year.
Figure 11: \textbf{Expected investments over the life-cycle for different income levels}. The figure shows the initial expectations of the investments in bonds, stocks, and housing units over the life-cycle. The medium-dark gray curves are for the benchmark initial income \( y = 20,000 \) USD per year, the black curves are for a lower initial income of \( y = 10,000 \) USD per year, whereas the light gray curves are for a higher initial income of \( y = 30,000 \) USD per year. For other parameters, the benchmark values in Table 1 are applied. The current short rate is set to the long-term level, \( r = \bar{r} \), the current house price is \( h = 250 \) USD per housing unit, and the current financial wealth is \( x = 20,000 \) USD.
Figure 12: Expected investments over the life-cycle for different income volatilities. The figure shows the initial expectations of the investments in bonds, stocks, and housing units over the life-cycle. The medium-dark gray curves are for the benchmark income volatility of $\bar{\sigma}_Y = 0.075$, the black curves are for a lower volatility of $\bar{\sigma}_Y = 0.05$, whereas the light gray curves are for a higher volatility of $\bar{\sigma}_Y = 0.10$. For other parameters, the benchmark values in Table 1 are applied. The current short rate is set to the long-term level, $r = \bar{r}$, the current house price is $h = 250$ USD per housing unit, the current financial wealth is $x = 20,000$ USD, and the current income is $y = 20,000$ USD per year.
Figure 13: Expected investments over the life-cycle for age-dependent income volatility. The figure shows the initial expectations of the investments in bonds, stocks, and housing units over the life-cycle. The solid curves are for the benchmark case in which the income volatility is constant and equal to 7.5% over the 30 year working period and then drops to and stays at zero. The dashed curves are for the case in which the income volatility is initially 15% and then drops linearly to zero at retirement and stays at zero. For other parameters, the benchmark values in Table 1 are applied. The current short rate is set to the long-term level, \( r = \bar{r} \), the current house price is \( h = 250 \) USD per housing unit, the current financial wealth is \( x = 20,000 \) USD, and the current income is \( y = 20,000 \) USD per year.
Figure 14: Expected income over the life-cycle for three educational groups. The figure shows the initial expectations of annual income over the life-cycle for three different educational groups. Individuals are initially 25 years, retire at age 65, and live until age 85. The expected income growth is a polynomial in age with the education-specific coefficients listed in Table 2. The initial annual income is fixed at $y = 20,000$ USD per year and the initial interest rate is set to the long-term level, $r = \bar{r}$. The dark gray curve is for an individual with no high school education, the medium gray curve for an individual with a high school education, and the light gray curve for an individual with a college education. The dashed curve is for the benchmark case in which the expected income growth is independent of age.
Figure 15: Expected wealth over the life-cycle for three educational groups. The figure shows the initial expectations of total wealth, financial wealth, and human wealth over the life-cycle for three different educational groups. Individuals are initially 25 years, retire at age 65, and live until age 85. The expected income growth is a polynomial in age with the education-specific coefficients listed in Table 2. For other parameters, the benchmark values in Table 1 are applied. The current short rate is set to the long-term level, \( r = \bar{r} \), the current house price is \( h = 250 \) USD per housing unit, the current financial wealth is \( x = 20,000 \) USD, and the current income is \( y = 20,000 \) USD per year. The light gray curves are for an individual with no high school education, the medium gray curves for an individual with a high school education, and the dark gray curves for an individual with a college education. The black curves are for the benchmark case in which the expected income growth is independent of age.
Figure 16: **Expected investments over the life-cycle for three educational groups.** The figure shows the initial expectations of the investments in bonds, stocks, and housing units over the life-cycle. Individuals are initially 25 years, retire at age 65, and live until age 85. The expected income growth is a polynomial in age with the education-specific coefficients listed in Table 2. For other parameters, the benchmark values in Table 1 are applied. The current short rate is set to the long-term level, \( r = \bar{r} \), the current house price is \( h = 250 \) USD per housing unit, the current financial wealth is \( x = 20,000 \) USD, and the current income is \( y = 20,000 \) USD per year. The light gray curves are for an individual with no high school education, the medium gray curves for an individual with a high school education, and the dark gray curves for an individual with a college education.
Figure 17: Consumption of and investment in housing over the life-cycle for three educational groups. The figure shows the initial expectations of the number of housing units consumed and the number of housing units invested in physically or via financial contracts over the life-cycle. Individuals are initially 25 years, retire at age 65, and live until age 85. The expected income growth is a polynomial in age with the education-specific coefficients listed in Table 2. For other parameters, the benchmark values in Table 1 are applied. The current short rate is set to the long-term level, $r = \bar{r}$, the current house price is $h = 250$ USD per housing unit, the current financial wealth is $x = 20,000$ USD, and the current income is $y = 20,000$ USD per year. The light gray curves are for an individual with no high school education, the medium gray curves for an individual with a high school education, and the dark gray curves for an individual with a college education.
Figure 18: **Expected investments over the life-cycle for different correlations.** The figure shows the initial expectations of the investments in bonds, stocks, and housing units over the life-cycle. The medium-dark gray curves are for the benchmark case in which the house-income correlation is $\rho_{HY} = 0.5723$ and the house-stock correlation is $\rho_{HS} = 0.5$. The black curves are for a lower house-income correlation of $\rho_{HY} = 0.5$ and a house-stock correlation of $\rho_{HS} = 0.5723$. The light gray curves are for a higher house-income correlation of $\rho_{HY} = 0.65$ and a house-stock correlation of $\rho_{HS} = 0.3937$. For other parameters, the benchmark values in Table 1 are applied. The current short rate is set to the long-term level, $r = \bar{r}$, the current house price is $h = 250$ USD per housing unit, the current financial wealth is $x = 20,000$ USD, and the current income is $y = 20,000$ USD per year.
Figure 19: **Expected investments over the life-cycle for different house-income correlations.**

The figure shows the initial expectations of the investments in bonds, stocks, and housing units over the life-cycle. The medium-dark gray curves are for the benchmark case in which the house-income correlation is $\rho_{HY} = 0.5723$, the black curves are for a lower house-income correlation of $\rho_{HY} = 0.5$, and the light gray curves are for a higher house-income correlation of $\rho_{HY} = 0.65$. The house-stock correlation is fixed at $0.3937$. For other parameters, the benchmark values in Table 1 are applied. The income is only spanned when $\rho_{HY} = 0.65$, and in that case the strategies used are the truly optimal. For $\rho_{HY}$ equal to 0.5 or 0.5723, the strategies are only near-optimal as explained in the text. The current short rate is set to the long-term level, $r = \bar{r}$, the current house price is $h = 250$ USD per housing unit, the current financial wealth is $x = 20,000$ USD, and the current income is $y = 20,000$ USD per year.
Figure 20: **The utility loss due to a fixed level of housing consumption.** The figure shows how the utility loss measured by the $\ell$ defined in (5.2) with $\tilde{J} = J^{dc}$ varies with an assumed fixed level of housing consumption throughout life. The results are generated with the benchmark parameters in Table 1. The current short rate is set to the long-term level, $r = \bar{r}$, the current house price is $h = 250$ USD per housing unit, the current financial wealth is $x = 20,000$ USD, and the current income is $y = 20,000$ USD per year.
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<tr>
<td>$\zeta_B$ -1.1358</td>
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<td>$\rho_{SB}$ 0</td>
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**Stock**

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<tbody>
<tr>
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<td>$\rho_{HY}$ 0.5723</td>
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Table 1: **Benchmark parameter values.** Within each group of parameters, the exogenous parameters are stated above the dashed horizontal line, whereas the parameters below the dashed line are derived from the exogenous parameters.
<table>
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Table 2: **Constants in the income growth rate polynomials.** The table lists the relevant coefficients of the estimated 3rd order polynomials for the income growth rate over the life-cycle, cf. Table 2 in Cocco, Gomes, and Maenhout (2005).
Table 3: Loss due to infrequent adjustments of housing consumption and investment. The current short rate is set to the long-term level, $r_0 = \bar{r}$, the current house price is $h = 250$ USD per housing unit, and the current tangible wealth is set to $x = 20,000$ USD. The current annual income is set to $y = 10,000$ USD in the left part of the table and $y = 20,000$ USD (the benchmark value) in the right part. We assume the benchmark parameter values listed in Table 1. The certainty equivalent wealth loss is defined in (5.2), where $\hat{J}$ is computed using Monte Carlo simulations. “Frequent” means that the control is updated at the simulation frequency, i.e., 250 times per year. The results are based on 10,000 simulated paths.

<table>
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<td>5 years</td>
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<td>Infrequent $\varphi_C$, frequent $\varphi_I$</td>
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