Bond Durations: Corporates vs. Treasuries[a]

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Bond Durations: Corporates vs. Treasuries

ABSTRACT: We compare the durations (the percentage price sensitivity with respect to the default-free short rate) of corporate and Treasury bonds in the reduced-form, intensity-based credit risk modeling framework. In a frequently used intensity-based model for corporate bond valuation we provide an example showing that, given the parameter estimates found in empirical studies, the duration of a corporate coupon bond may very well be larger than the duration of a similar Treasury bond. This finding contrasts with conclusions of previous studies. In a general, intensity-based recovery of market value framework we provide a simple sufficient condition for when the duration of a corporate bond will be smaller than that of a similar Treasury bond. We also provide an upper bound on the duration of the corporate coupon bond.

KEYWORDS: duration, interest rate risk, default risk, intensity models

JEL-CLASSIFICATION: E43, G12

1 Introduction

The duration of an asset is a measure of its interest rate risk. Although duration has a long history, it is still an important and widely used tool in the risk management of portfolios of interest rate sensitive assets. Most papers studying duration focus on default-free (Treasury) bonds, but for the many portfolio managers also investing in defaultable (corporate) bonds it is important to understand the sensitivity of defaultable bonds to interest rate changes. The few existing papers addressing the duration of corporate bonds either derive durations from relatively simple firm-value based models or estimate the empirical relation between changes in the prices of corporate bonds and changes in interest rates. Furthermore, the statements on the duration of a corporate bond seem to be conflicting: Chance (1990) states that

> "...defaultable bonds have durations lower than their maturities and, thus, are less sensitive to interest rates than their default-free counterparts ..."

whereas, in a more recent paper, Jacoby (2003) finds that

> "...the duration of a non-callable corporate bond is longer than its Macaulay counterpart ..."

In this paper we define the duration of an asset as the percentage change in the asset price caused by a marginal increase in the default-free short-term interest rate. We compare the duration of a default-free bond and the duration of the equivalent defaultable bond, i.e. the defaultable bond promising the same payments as the default-free bond. Throughout the paper we apply the modern reduced-form framework for corporate bond valuation with the recovery-of-market specification of Duffie and Singleton (1999).

First, in a setting with default-free interest rates according to the Vasicek model, a constant loss rate given default, and a default intensity which is affine in the short rate, we show that the duration of the defaultable coupon bond is smaller than that of the equivalent default-free bond unless the default intensity is sufficiently positively correlated with the default-free short-term interest rate. Several empirical studies provide information on the magnitude of this sensitivity, e.g. Longstaff and Schwartz (1995), Duffee (1998), Jarrow and Yildirim (2002), and Bakshi, Madan, and Zhang (2006). Given the range of parameter estimates reported in those studies, the duration of a corporate coupon bond can either be greater than or smaller than the duration of a similar Treasury bond.

Second, for zero-coupon bonds we note that the duration of a corporate bond is identical to that of a default-free bond if default and recovery risk is independent of default-free interest rates. For the case when the default intensity is an affine function of the default-free short rate and the loss rate is constant we provide a relatively simple equation linking the durations of the corporate and the default-free bonds. When default-free rates are described by the Ho-Lee or the Vasicek model, the duration of the corporate zero-coupon bond equals the duration of the equivalent default-free bond multiplied by a factor $1+L\Lambda_1$, where L is the loss rate given default and Λ_1 is the sensitivity of the default intensity with respect to the default-free short rate. For the Cox-Ingersoll-Ross model of default-free interest rates, this relation is shown to be a very good approximation. In particular, when the default intensity is decreasing (increasing) in the interest rate level, the corporate zero-coupon bond has a lower (higher) duration than the default-free bond.

Third, under weak assumptions on the default-free interest rates, the default intensity, and the loss rate in case of default, we show that the duration of a corporate coupon bond is smaller than the duration of the equivalent Treasury bond, whenever the same is true for all the zero-coupon bonds embedded in the coupon bond. In particular, this is true in an affine model where the default intensity is decreasing in the default-free short rate. In the same setting we provide an upper bound of the duration of the defaultable bond in terms of the duration of the equivalent default-free bond, the loss rate, and the short rate sensitivity of the default intensity. The upper bound is tight for relatively short-maturity bonds. We also show that even if corporate zero-coupon bonds have higher durations than their default-free counterparts, the duration of a corporate coupon bond may be smaller than that of the equivalent default-free coupon bond.

Let us briefly review the related literature. In the very simple Merton (1974) setting for corporate debt valuation, Chance (1990) shows that the duration of a defaultable zero-coupon bond is smaller than the duration of the similar default-free zero-coupon bond. Fooladi, Roberts, and Skinner (1997) define and study a duration-style measure in a specific pricing model different from the mainstream models applied today for the pricing of defaultable claims. Babbel, Merrill, and Panning (1997) set up a pricing model with the default-free short rate and the value of the issuing firm as state variables. They calibrate the model to data and derive an estimated relation between corporate bond prices and default-free interest rates. Using that relation they conclude that default risk shortens the duration. In a similar setting Acharya and Carpenter (2002) endogenize the default decision by the issuer and study, among other things, how the duration of the corporate bond depends on the firm value. They conclude that default risk reduces the duration of a bond. Longstaff and Schwartz (1995) make similar conclusions about duration in their valuation model for corporate bonds. In fact they also argue that the duration of a corporate bond may very well be negative.[1] While all these papers thus agree that corporate bonds have smaller durations than default-free bonds, Jacoby (2003) arrives at the opposite conclusion, cf. the quote in the beginning of this introduction. We show that whether the duration of a corporate bond is smaller or larger than the duration of the equivalent Treasury bond is mainly determined by the short rate sensitivity of the default intensity and that both conclusions are possible in empirically relevant settings.

[1]In the firm-value framework an increase in default-free interest rate has two opposing effects on the value of corporate debt. First, it will decrease the present value of the future cash flow to the debt. Second, it will increase the (risk-neutral) expected rate of return on the assets of the issuing firm and, hence, lower the default probability and increase the expected cash flow to the debt. If the last term dominates, the corporate bond price will be increasing in the default-free short rate, corresponding to having a negative duration.

The rest of this paper is organized as follows. In Section 2 we define duration formally and briefly review the intensity-based, recovery-of-market framework for the valuation of defaultable bonds. Section 3 specifies a concrete model in which we compute and compare the durations of corporate and Treasury bonds. Section 4 provides the formal analysis of corporate and Treasury bond durations in a general setting. The implications of our findings for risk management are addressed in Section 5. Finally, Section 6 concludes.

2 The Basic Framework and the Concept of Duration

We consider an arbitrage-free financial market allowing the existence of a risk-neutral probability measure. The market has an instantaneously risk-free asset (a money market account) with r_t denoting the continuously compounded short-term default-free interest rate at time t (the short rate). We define the duration of any asset as minus the percentage price sensitivity with respect to the short rate, i.e. if V_t denotes the time t value of the asset, the duration is defined as

$$D_t^V = -\frac{\partial V_t}{\partial r}\frac{1}{V_t}, \tag{1}$$

if the value V_t is differentiable with respect to the short rate r, as we will assume is the case in the following.

The duration of a bond was originally defined as minus the derivative of its price with respect to its own yield-to-maturity, divided by the price. However, it is well-known that the application of that duration in risk management requires a flat zero-coupon yield curve which can only change in form of parallel shifts and that this is incompatible with dynamic arbitrage-free term structure models; see, e.g., Ingersoll, Skelton, and Weil (1978) and Cox, Ingersoll, and Ross (1979). Moreover, the original definition only makes sense for bonds, not for other interest rate sensitive assets. In the context of dynamic term structure models, the price sensitivity to the default-free short rate is a much more appropriate and tractable measure of interest rate risk and it is well-defined for a broad set of assets. In the popular one-factor term structure models of Vasicek (1977) and Cox, Ingersoll, and Ross (1985), the short rate captures the variations in the full default-free yield curve and is therefore the natural single variable to use when measuring price sensitivities. Since in those models the yield of any default-free bond is affine in the short rate, one can equivalently measure price sensitivities towards changes in some specific yield, which is more in line with the traditional definition of duration. Other models add one or more extra factors to the short rate and, obviously, the price sensitivities towards changes in those other factors are also relevant for risk measurement and management. For any other factor we can define a factor duration similar to the short rate duration in (1) and all the results we derive for short rate durations will also be valid for those factor durations if, of course, the relevant assumptions are satisfied. For simplicity and concreteness we focus on the duration reflecting the price sensitivity towards short rate variations.

We want to compare the duration of a default-free coupon bond to the duration of a defaultable coupon bond promising payments identical to those of the default-free bond.

All the bonds we consider are assumed to have a face value of 1. We focus on fixed-rate bullet bonds maturing at time t_n and discrete, periodic coupon payments of q at time $t_1, t_2, \ldots, t_n$, where $t_1 < t_2 < \cdots < t_n$. All conclusions are also valid for bonds promising a fixed continuous coupon.

First, consider default-free bonds. The time t price of the default-free zero-coupon bond maturing at time $t_n \geq t$ is $\bar{P}_t^{t_n} = \mathrm{E}_t[e^{-\int_t^{t_n} r_u \, du}]$, where $\mathrm{E}_t[\;]$ is the expectation under some fixed risk-neutral probability measure conditional on time t information. The time t price of the default-free coupon bond is then

$$\bar{P}_t^{q,t_n} = q \sum_{t_j > t} \bar{P}_t^{t_j} + \bar{P}_t^{t_n},$$

where the sum is over all $j \in \{1, \ldots, n\}$ with $t_j > t$, i.e. all future coupon payment dates. Let

$$\bar{D}_t^s = -\frac{\partial \bar{P}_t^s}{\partial r} \frac{1}{\bar{P}_t^s}$$

denote the duration of the default-free zero-coupon bond maturing at time s. Then the duration of the default-free coupon bond, $\bar{D}_t^{q,t_n}$, is

$$\begin{aligned} \bar{D}_t^{q,t_n} &= -\frac{\partial \bar{P}_t^{q,t_n}}{\partial r} \frac{1}{\bar{P}_t^{q,t_n}} \\ &= \left(q \sum_{t_j > t} \bar{P}_t^{t_j} \bar{D}_t^{t_j} + \bar{P}_t^{t_n} \bar{D}_t^{t_n} \right) \frac{1}{\bar{P}_t^{q,t_n}} \\ &= \sum_{t_j > t} \frac{q \bar{P}_t^{t_j}}{\bar{P}_t^{q,t_n}} \bar{D}_t^{t_j} + \frac{\bar{P}_t^{t_n}}{\bar{P}_t^{q,t_n}} \bar{D}_t^{t_n}, \end{aligned} \tag{2}$$

i.e. a weighted average of the durations of the default-free zero-coupon bonds maturing at the different payment dates of the coupon bond.

Next, consider defaultable bonds. The default time of the bond issuer is represented by the stopping time τ. We apply the recovery of market value specification so that the payment received at the time of default equals $(1 - l_\tau)$ times the value of the same bond immediately before the default so that l_τ is the percentage loss at default. Let $\lambda = (\lambda_t)$ denote the intensity process of the default time. Following Theorem 1 and the subsequent discussion in Duffie and Singleton (1999), the price of a defaultable coupon bond is

$$P_t^{q,t_n} = q \sum_{t_j > t} P_t^{t_j} + P_t^{t_n},$$

where

$$P_t^{t_j} = \mathrm{E}_t \left[e^{-\int_t^{t_j} (r_u + \lambda_u l_u) \, du} \right], \quad j = 1, \ldots, n,$$

is the time t price of a hypothetical defaultable zero-coupon bond maturing at t_j with the same proportional loss rate process $l = (l_t)$ as for the coupon bond. Consequently, the duration of the defaultable coupon bond is a weighted average of the durations of these zero-coupon defaultable bonds:

$$D_t^{q,t_n} = \sum_{t_j>t} \frac{qP_t^{t_j}}{P_t^{q,t_n}} D_t^{t_j} + \frac{P_t^{t_n}}{P_t^{q,t_n}} D_t^{t_n}, \tag{3}$$

where $D_t^{t_j} = -\frac{\partial P_t^{t_j}}{\partial r}/P_t^{t_j}$ is the duration of the defaultable zero-coupon bond maturing at time t_j.

Note that it is not clear whether the duration of the default-free bond in (2) or the duration of the defaultable bond in (3) is greater. This can already be seen in the special case when interest rate risk is independent of default risk and recovery risk. Under this assumption, one can easily show that the durations of default-free and defaultable zero-coupon bonds coincide, i.e. $\bar{D}_t^{t_j} = D_t^{t_j}$ (see Proposition 1 below). Hence, the values of the weights $\bar{P}_t^{t_j}/\bar{P}_t^{q,t_n}$ and $P_t^{t_j}/P_t^{q,t_n}$ in the representations of the durations play decisive roles. Since corporate bond prices are decreasing in the degree of default risk, we have both $\bar{P}_t^{t_j} > P_t^{t_j}$ and $\bar{P}_t^{q,t_n} > P_t^{q,t_n}$ implying that the overall impact of default risk on duration remains open. Analyzing this point in detail is the subject of our paper.

3 An Affine Example

Assume that the default-free interest rates can be described by the Vasicek (1977) model. In particular, the risk-neutral dynamics of the default-free short rate r_t is

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma_r\,dW_t, \tag{4}$$

and the price of a default-free zero-coupon bond is

$$\bar{P}_t^T = e^{-\bar{A}(T-t)-\bar{B}(T-t)r_t}, \tag{5}$$

where

$$\bar{B}(\tau) = \frac{1}{\kappa}\left(1 - e^{-\kappa\tau}\right),$$
$$\bar{A}(\tau) = \frac{1}{\kappa}\left(\theta - \frac{\sigma_r^2}{2\kappa}\right)\left(\tau - \bar{B}(\tau)\right) + \frac{\sigma_r^2}{4\kappa}\bar{B}(\tau)^2.$$

The duration of this bond is $\bar{D}_t^T = \bar{B}(T-t)$, which is concavely increasing in time to maturity.

Assume that recovery of market value applies and the default risk adjusted short rate is affine in the default-free short rate,

$$R_t \equiv r_t + \lambda_t l_t = k_0 + k_1 r_t. \tag{6}$$

Then it can be shown that the corporate zero-coupon bond price $P_t^T = \mathrm{E}_t[\exp\{-\int_t^T R_u\,du\}]$ becomes

$$P_t^T = e^{f(T-t)}\left(\bar{P}_t^T\right)^{k_1}, \tag{7}$$

where

$$f(\tau) = -k_0\tau + \frac{\sigma_r^2 k_1(k_1-1)}{2\kappa^2}\left(\tau - \bar{B}(\tau) - \frac{\kappa}{2}\bar{B}(\tau)^2\right).$$

The duration of the corporate zero-coupon bond is

$$D_t^T = k_1\bar{D}_t^T = k_1\bar{B}(T-t), \tag{8}$$

which is smaller than the duration of the equivalent default-free zero-coupon bond if, and only if, $k_1 < 1$.

Equation (6) is satisfied in two cases:

(i) Constant loss rate, $l_t = L \geq 0$, and affine default intensity, $\lambda_t = \Lambda_0 + \Lambda_1 r_t$. Then $k_0 = \Lambda_0 L$, $k_1 = 1 + \Lambda_1 L$. Hence, the duration of the corporate zero-coupon bond will be smaller [larger] than the duration of the default-free zero-coupon bond of the same maturity if $\Lambda_1 < 0$ [if $\Lambda_1 > 0$]. The duration of the corporate bond will be negative if $\Lambda_1 < -1/L$, which is theoretically possible. Also note that the sign of the duration of the corporate bond does not depend on the *level* of the default probability (as the discussion in Longstaff and Schwartz (1995) suggests) but on the *interest rate sensitivity* of the default probability.

(ii) Constant default intensity, $\lambda_t = \Lambda > 0$, and affine loss rate, $l_t = L_0 + L_1 r_t$. Then $k_0 = \Lambda L_0$, $k_1 = 1 + \Lambda L_1$, and the duration of the corporate bond will be smaller [larger] than the duration of the default-free bond if $L_1 < 0$ [if $L_1 > 0$].

The empirical evidence on the key parameters is mixed and leads to different conclusions on the comparison of durations. Using data on U.S. Treasury and corporate bond prices over the period 1989-1998, Bakshi, Madan, and Zhang (2006) estimate several specifications of the default risk adjusted short rate R_t in the Duffie-Singleton framework. For a model with $R_t = k_0 + k_1 r_t$, consistent with our assumption above, their estimate of k_1 is 1.018 for BBB-rated bonds (implying a higher duration of the corporate bond) and 0.985 for A-rated bonds (implying a lower duration of the corporate bond). Adding a firm-specific distress variable S_t so that $R_t = k_0 + k_1 r_t + k_2 S_t$, they estimate k_1 to be below one, namely in the range $[0.767, 0.910]$ for BBB bonds and in $[0.902, 0.966]$ for A bonds with the estimates depending on the proxy used for financial distress.

Jarrow and Yildirim (2002) assume a generalized Vasicek model, a constant loss rate, and a default intensity affine in the default-free short rate. They estimate the parameters using corporate default swap quotes for 22 individual companies. Their company-specific estimates of Λ_1 are positive, ranging from 1.3 to 26.9 basis points, implying that the duration of a zero-coupon bond issued by any of these companies would be greater than the duration of a default-free zero-coupon bond of the same maturity.

The yields of the default-free and the corporate zero-coupon bonds are

$$\bar{y}_t^{t+\tau} = \frac{\bar{A}(\tau)}{\tau} + \frac{\bar{B}(\tau)}{\tau} r_t, \qquad y_t^{t+\tau} = k_1 \bar{y}_t^{t+\tau} - k_1 \frac{f(\tau)}{\tau},$$

respectively, so that the yield spread becomes

$$y_t^{t+\tau} - \bar{y}_t^{t+\tau} = (k_1 - 1)\bar{y}_t^{t+\tau} - k_1\frac{f(\tau)}{\tau} = (k_1 - 1)\frac{\bar{A}(\tau)}{\tau} - k_1\frac{f(\tau)}{\tau} + (k_1 - 1)\frac{\bar{B}(\tau)}{\tau}r_t.$$

Empirical studies of Longstaff and Schwartz (1995), Duffee (1998), and Papageorgiou and Skinner (2006) conclude that yield spreads are generally decreasing in default-free yields. Within the current setting, this will be the case if $k_1 < 1$, which is exactly when the duration of any corporate zero-coupon bond is smaller than the duration of the default-free bond of the identical maturity.

Next, consider the duration of coupon bonds in this model. The duration of a default-free coupon bond is

$$\bar{D}_t^{q,t_n} = \sum_{t_j > t} \bar{a}_t^{t_j} \bar{D}(t_j - t) = \sum_{t_j > t} \bar{a}_t^{t_j} \bar{B}(t_j - t),$$

where

$$\bar{a}_t^{t_j} = \begin{cases} q\frac{\bar{P}_t^{t_j}}{\bar{P}_t^{q,t_n}} & \text{for } t_j < t_n, \\ (1+q)\frac{\bar{P}_t^{t_j}}{\bar{P}_t^{q,t_n}} & \text{for } t_j = t_n. \end{cases}$$

For a similar corporate coupon bond, the duration is

$$D_t^{q,t_n} = \sum_{t_j > t} a_t^{t_j} D_t^{t_j} = k_1 \sum_{t_j > t} a_t^{t_j} \bar{D}_t^{t_j} = k_1 \sum_{t_j > t} a_t^{t_j} \bar{B}(t_j - t),$$

where

$$a_t^{t_j} = \begin{cases} q\frac{P_t^{t_j}}{P_t^{q,t_n}} & \text{for } t_j < t_n, \\ (1+q)\frac{P_t^{t_j}}{P_t^{q,t_n}} & \text{for } t_j = t_n. \end{cases}$$

The relation between the durations of the two coupon bonds depends on the value of k_1 but also on the weights $\bar{a}_t^{t_j}$ and $a_t^{t_j}$. Note that $\sum_{j=1}^n \bar{a}_t^{t_j} = 1 = \sum_{j=1}^n a_t^{t_j}$ so if $\bar{a}_t^{t_j} > a_t^{t_j}$ for one payment date t_j, the opposite inequality holds for for another payment date t_k. For the case where $k_1 = 1$ we have $f(\tau) = -k_0\tau$ and it can be shown that

$$\frac{\bar{a}_t^{t_j}}{a_t^{t_j}} = \frac{\sum_{t_k > t} q e^{k_0(t_j - t_k)} \bar{P}_t^{t_k} + e^{k_0(t_j - t_n)} \bar{P}_t^{t_n}}{\sum_{t_k > t} q \bar{P}_t^{t_k} + \bar{P}_t^{t_n}}.$$

Since $k_0 > 0$, this ratio will be greater than 1 for the last payment date ($j = n$) and smaller than one for the first payment date ($j = 1$). Recalling that the function $\bar{B}$ is increasing, this means that higher weights will be applied to high values of the $\bar{B}$-function in the computation of the duration of the default-free bond than in the computation of the duration of the corporate bond. Consequently, the duration of the default-free coupon bond will be higher than the duration of the corporate coupon bond.

For concreteness assume that version (i) of the model applies, i.e. that the loss rate is given by a constant L and the default intensity is $\lambda_t = \Lambda_0 + \Lambda_1 r_t$. We apply the parameter values listed in Table 1. In particular the current short rate is 4% and the yield curve is upward-sloping with an asymptotic long-term zero-coupon yield of 5%. We consider

10-year bullet bonds with semi-annual coupons of 3%. For the corporate bond the loss in case of default is assumed to be 40%. The fixed part of the default intensity is $\Lambda_0 = 0.025$ so that $k_0 = 0.01$. We vary Λ_1 between -0.5 and 0.5 corresponding to variations in default intensities (at the current short rate) between 0.005 and 0.045 and variations in k_1 between 0.8 and 1.2. Figure 1 shows yield spread curves for different values of Λ_1.

[Table 1 about here.]

[Figure 1 about here.]

With the assumed parameters the duration of the 10-year Treasury coupon bond is 4.3099. Figure 2 shows how the duration of the 10-year corporate coupon bond varies with Λ_1. For $\Lambda_1 < 0$, i.e. $k_1 < 1$, the duration of a corporate zero-coupon bond is smaller than the duration of the similar default-free zero-coupon bond for any maturity, cf. (8), and we see from the figure that the duration of the corporate coupon bond is then also smaller than the duration of the Treasury coupon bond. This conclusion applies more generally as will be shown in Theorem 1 below: under a given condition, the duration of a corporate coupon bond is smaller than that of a similar Treasury coupon bond whenever the same is true for zero-coupon bonds. For $\Lambda_1 = 0$, i.e. $k_1 = 1$, the durations of a corporate and a default-free zero-coupon bond of the same maturity are identical. Nevertheless, as explained above, the duration of the 10-year corporate coupon bond (4.2663) is smaller than the duration of the similar default-free bond. In fact, this is also true for slightly positive values of Λ_1 (up to approximately 0.027), which shows that even if the durations of corporate zero-coupon bonds are *higher* than the durations of default-free zero-coupon bonds, the duration of a corporate coupon bond may be *smaller* than the duration of a default-free coupon bond. For values of Λ_1 higher than 0.027 (i.e. k_1 higher than approximately 1.011), the durations of both corporate zero-coupon and corporate coupon bonds are higher than those of their default-free counterparts. Given the empirical parameter estimates reported above, this may very well be the case for a number of corporate bonds.

[Figure 2 about here.]

4 The General Analysis

In this section, we first analyze a more general, possibly non-affine model. This is much more involved since closed-form solutions for the durations are usually not available. Subsequently, we show how our results simplify for affine models. We consider zero-coupon bonds in Section 4.1 and coupon bonds in Section 4.2.

4.1 Duration of Corporate Zero-Coupon Bonds

Let us start with a simplified problem: If interest rate risk is independent of default and recovery risk, then the price of a defaultable bond can be rewritten as

$$P_t^s = \mathrm{E}_t\left[e^{-\int_t^s (r_u+\lambda_u l_u)\,du}\right] = \mathrm{E}_t\left[e^{-\int_t^s r_u\,du}\right]\mathrm{E}_t\left[e^{-\int_t^s \lambda_u l_u\,du}\right] = \bar{P}_t^s \mathrm{E}_t\left[e^{-\int_t^s \lambda_u l_u\,du}\right]. \quad (9)$$

This immediately leads to the following result.

Proposition 1 (Independence) *If interest rate risk is independent of default risk and recovery risk, then the durations of a default-free zero coupon bond and the corporate zero-coupon bond with zero-recovery coincide, i.e.* $\bar{D}_t^s = D_t^s$.

Without the independence assumption this result is generally not valid, cf. the example of the previous section. To gain further insights, throughout this subsection we place our considerations in a Cox process framework where the default intensity is driven by a number of state variables. The dynamics of the short rate is assumed to be given by the stochastic differential equation (SDE)

$$dr_s = \alpha(s, r_s)\, ds + \beta(s, r_s)\, dW_s, \quad r_t = \bar{r}, \tag{10}$$

where W is a (possibly) multi-dimensional Brownian motion. It is assumed that (10) has a unique solution. Recall that the time t prices of a default-free and a defaultable zero-coupon bond with maturities s are given by

$$\bar{P}_t^s = \mathrm{E}_t\left[e^{-\int_t^s r_u\, du}\right] \tag{11}$$

and

$$P_t^s = \mathrm{E}_t\left[e^{-\int_t^s (r_u + L\lambda_u)\, du}\right], \tag{12}$$

respectively, where λ denotes the default intensity and L denotes the loss rate which, for simplicity, is assumed constant in this subsection. Besides, we assume that the intensity λ is a function of the short rate (and maybe other state variables) with a well-defined derivative denoted by λ_r. For notational convenience, we suppress the dependencies on the other state variables.

Looking at the bond price expressions (11) and (12), it is clear that the derivatives of the bond price with respect to the current default-free short rate will depend on the sensitivity of the future short rates with respect to the current short rate. For a fixed t, define $y_u \equiv \frac{\partial}{\partial r_t} r_u$. Under mild regularity conditions, the dynamics of y_u is given by

$$dy_s = y_s[\alpha_r(s, r_s)\, ds + \beta_r(s, r_s)\, dW_s], \quad y_t = 1,$$

where α_r and β_r denote the partial derivatives of α and β with respect to r; cf. Protter (2005, pp. 311ff). Applying Ito's lemma, one obtains the explicit solution

$$y_u = \exp\left(\int_t^u [\alpha_r(s, r(s)) - 0.5\beta_r(s, r_s)^2]\, ds + \int_t^u \beta_r(s, r_s)\, dW_s\right) \geq 0.$$

Taking derivatives of the zero-coupon bond prices with respect to the initial short rate $r = r_t$ gives

$$-\frac{\partial \bar{P}_t^s}{\partial r} = \mathrm{E}_t\left[e^{-\int_t^s r_u\, du} \int_t^s y_u\, du\right] = \bar{P}_t^s \int_t^s \mathrm{E}_t^{Q^s}[y_u]\, du$$

and

$$-\frac{\partial P_t^s}{\partial r} = \mathrm{E}_t\left[e^{-\int_t^s (r_u + L\lambda_u)\, du} \int_t^s y_u(1 + L\lambda_r(r_u))\, du\right], \tag{13}$$

where Q^s denotes the s-forward measure. Therefore, the duration of a default-free zero is given by

$$\bar{D}_t^s = -\frac{\partial \bar{P}_t^s}{\partial r}\frac{1}{\bar{P}_t^s} = \int_t^s \mathrm{E}_t^{Q^s}\left[y_u\right] du \geq 0.$$

Note that, on the one hand, the derivative of the corporate bond price $\partial P_t^s/\partial r$ is negative if $\lambda_r > -1/L$ and therefore, in particular, when $\lambda_r > -1$. On the other hand, the default intensity is positive, i.e. $\lambda \geq 0$, and $y \geq 0$. Hence, for $\lambda_r < 0$ we conclude

$$-\frac{\partial \bar{P}_t^s}{\partial r} \geq -\frac{\partial P_t^s}{\partial r}. \tag{14}$$

If, however, $\lambda_r > 0$, then the relation need not hold. Consequently, the dependency of the intensity λ on the short rate r plays a crucial role again. To calculate the durations of both bonds, we need to divide both derivatives by the respective bond price. If (14) holds, it is not clear which duration is greater because $\bar{P}_t^s \geq P_t^s$. Only if (14) is violated, then the duration of the corporate bond is always greater than the duration of the default-free bond, but this is not the typical situation.

To gain further insights, we assume for the rest of this subsection that the intensity is an affine function of the short rate r, i.e.

$$\lambda_t = \Lambda_0 + \Lambda_1 r_t \tag{15}$$

with constants Λ_0 and Λ_1. This assumption is also made in Bakshi, Madan, and Zhang (2006) as well as in Jarrow and Yildirim (2002) and can be interpreted as a first-order approximation. Note that higher order terms could be included as well. Then we obtain the following result which is proved in the Appendix.

Proposition 2 (Affine intensity, constant loss rate) *(i) If the intensity is affine of the form (15), then*

$$D_t^s = (1 + L\Lambda_1)\big\{\bar{D}_t^s + C_t^s\big\}, \tag{16}$$

where

$$C_t^s = \frac{\mathrm{Cov}_t\left[e^{-\int_t^s r_u\,du}\int_t^s y_u\,du, e^{-\Lambda_1\int_t^s r_u\,du}\right] - \bar{D}_t^s \mathrm{Cov}_t\left[e^{-\int_t^s r_u\,du}, e^{-\Lambda_1\int_t^s r_u\,du}\right]}{e^{\Lambda_0(s-t)}P_t^s} \tag{17}$$

(ii) If the derivative of the short rate with respect to its initial value is deterministic, then we have $C_t^s = 0$ implying $D_t^s = (1 + L\Lambda_1)\bar{D}_t^s$.

C_t^s can be interpreted as a correction term stemming from the fact that, in general, the linearity of the intensity with respect to the short rate does not carry over to the corporate bond duration. The correction term vanishes only if the derivative y is deterministic, which is for instance the case in the Vasicek model and the Ho-Lee model.[2] In this case, the

[2] In the Ho-Lee model, the short rate dynamics are given by $dr_t = \alpha(t)dt + \beta dW_t$ where α is a deterministic function and β is a constant.

duration of the defaultable zero-coupon bond is greater than the duration of the default-free bond if $\Lambda_1 > 0$, i.e. if interest rate risk and default risk are positively correlated, and smaller if the opposite is true.

If the short rate and the intensity are uncorrelated, then $C_t^s = \Lambda_1 = 0$ and the durations of a default-free and a defaultable zero-coupon bond coincide. In this special case, the covariances of (17) are canceling out, which gives us a strong hint that usually the covariances go in opposite directions. Note further that both covariances are similar, since $\tilde{D}_t^s = \int_t^s y_u \, du$ can be interpreted as the pathwise duration of a Treasury zero-coupon bond. Therefore, both covariances can be rewritten as

$$\mathrm{Cov}_t\left[e^{-\int_t^s r_u\,du}\tilde{D}_t^s, e^{-\Lambda_1\int_t^s r_u\,du}\right] \quad \text{and} \quad \mathrm{Cov}_t\left[e^{-\int_t^s r_u\,du}\bar{D}_t^s, e^{-\Lambda_1\int_t^s r_u\,du}\right], \tag{18}$$

i.e. the only difference is that the first covariance involves the pathwise duration $\tilde{D}_t^s$, while the second covariance involves the (average) duration $\bar{D}_t^s$.

In order to assess the importance of the correction term we consider the tractable class of affine short rate processes (see, e.g., Duffie, Pan, and Singleton (2000))

$$dr_t = (K_0 + K_1 r_t)dt + \sqrt{H_0 + H_1 r_t}\, dW_t. \tag{19}$$

Standard arguments give that the price of the zero-coupon default-free bond is given by $P_t^s = \exp(-\bar{A}(s-t) - \bar{B}(s-t)r_t)$, where $\bar{B}(\tau) = \mathcal{B}(\tau; K_1, H_1)$ is a deterministic function satisfying the ordinary differential equation

$$\mathcal{B}'(\tau) = 1 + K_1\mathcal{B}(\tau) - 0.5H_1\mathcal{B}(\tau)^2 \tag{20}$$

with initial condition $\mathcal{B}(0) = 0$, and $\bar{A}(\tau) = \mathcal{A}(\tau; K_0, K_1, H_0, H_1)$ satisfies the equation $\mathcal{A}'(\tau) = K_0\mathcal{B}(\tau) - 0.5H_0\mathcal{B}(\tau)^2$ with $\mathcal{A}(0) = 0$. The duration is given by $\bar{D}_t^s = \bar{B}(s-t)$. For the corresponding corporate bond the following result is proved in the Appendix.

Proposition 3 (Affine intensity and short rate, constant loss rate) *Assume (15) and (19). Then the corporate zero-coupon price is given by* $P_t^s = e^{-A(s-t)-(1+L\Lambda_1)B(s-t)r_t}$, *where* $B(\tau) = \mathcal{B}(\tau; K_1, (1+L\Lambda_1)H_1)$. *Therefore, its duration equals*

$$D_t^s = (1 + L\Lambda_1)B(s-t)$$

and the correction term (17) is given by $C_t^s = B(s-t) - \bar{B}(s-t)$.

First note that $B(s-t)$ and $\bar{B}(s-t)$ do not coincide with the covariances in (18), i.e. this is an alternative decomposition of the correction term convenient in affine models. Also note that the correction term is zero if the short rate volatility is constant ($H_1 = 0$) implying $B = \bar{B}$. This result is in line with Proposition 2 and the generic example is the Vasicek model. If, however, $H_1 \neq 0$, then the correction term is non-zero and the generic example is the Cox-Ingersoll-Ross model, which we analyze next.

Example 1 The Cox-Ingersoll-Ross model follows by setting $K_0 = \kappa\theta$, $K_1 = -\kappa$, $H_0 = 0$, and $H_1 = \sigma^2$. In this model, $\bar{B}(s-t) = \mathcal{B}(s-t; \kappa, \sigma^2)$ and $B(s-t) = \mathcal{B}(s-t; \kappa, (1+L\Lambda_1)\sigma^2)$, where[3]

$$\mathcal{B}(\tau;\kappa,\sigma^2) = \frac{2}{\kappa + \gamma\coth(\gamma\tau/2)}, \quad \gamma = \sqrt{\kappa^2 + 2\sigma^2}.$$

The correction term is

$$C_t^s = B(s-t) - \bar{B}(s-t) = \frac{2}{\kappa + \hat{\gamma}\coth(\hat{\gamma}\tau/2)} - \frac{2}{\kappa + \gamma\coth(\gamma\tau/2)},$$

where $\hat{\gamma} = \sqrt{\kappa^2 + 2(1+L\Lambda_1)\sigma^2}$. The function $\gamma\coth(\gamma\tau/2)$ is increasing in γ and decreasing in τ. For $\Lambda_1 < 0$, we have $\hat{\gamma} < \gamma$ and, consequently, $C_t^s > 0$. Conversely for $\Lambda_1 > 0$. Moreover, the absolute value of C_t^s increases with time to maturity. Using the approximation $D_t^s \approx \hat{D}_t^s \equiv (1+L\Lambda_1)\bar{D}_t^s$, the absolute error is

$$\hat{D}_t^s - D_t^s = -(1+L\Lambda_1)C_t^s$$

and the relative error is

$$\frac{\hat{D}_t^s - D_t^s}{D_t^s} = -\frac{C_t^s}{\bar{D}_t^s + C_t^s} = \frac{\bar{B}(s-t)}{B(s-t)} - 1.$$

We assume that $\kappa = 0.1$, $\sigma = 0.05$, and vary Λ_1 between -0.05 and 0.05 and the time to maturity $s-t$ between 0 and 30. The absolute value of the correction term (and, thus, the approximation error) is highest for $s-t = 30$ for which the duration of the default-free bond is $\bar{D}_t^s = 8.74$. First assume zero recovery, i.e. $L = 1$. The minimum value of C_t^s is reached for $s-t = 30$ and $\Lambda_1 = 0.05$. For these parameter values, the duration of the defaultable bond is $D_t^s = 9.14$, the correction term equals -0.0331, the absolute approximation error is 0.035, and the relative approximation error is 0.381%. The maximum value of C_t^s is obtained for $s-t = 30$ and $\Lambda_1 = -0.05$ for which $C_t^s = 0.0336$, $D_t^s = 8.33$, the absolute error is -0.032, and the relative error is -0.383%. Numerical computations reveal that the relative errors are almost linear in the loss rate L, i.e. if L is reduced to the more realistic value of 0.5, then the errors are only half as big. These results indicate that the approximation $D_t^s \approx (1+L\Lambda_1)\bar{D}_t^s$ works very well. □

4.2 Duration of Corporate Coupon Bonds

This section analyzes the duration of coupon bonds. We provide strong results on durations of coupon bonds if the duration of a corporate zero-coupon bonds is smaller than the duration of the corresponding default-free zero-coupon bonds. In addition we present an upper bound on the duration of a corporate coupon bond. Our results are derived under the following two assumptions:

Assumption 1 *The duration of default-free zero-coupon bonds is non-decreasing in the maturity of the bond, i.e.* $\bar{D}_t^s$ *is non-decreasing in* s.

[3] $\coth(x) = (e^x + e^{-x})/(e^x - e^{-x})$ is the hyperbolic cotangent.

This assumption is satisfied in most dynamic term structure models, including the one-factor Vasicek and Cox-Ingersoll-Ross (CIR) models as well as the two-factor Hull-White extension of the Vasicek model.

Assumption 2 *The ratio of the corporate zero-coupon bond price to the equivalent default-free zero-coupon bond price is non-increasing in maturity, i.e.* $P_t^s/\bar{P}_t^s$ *is non-increasing in* s*.*

This assumption is satisfied if the price of a corporate zero-coupon bond decreases faster in its time to maturity than the price of a Treasury zero-coupon bond. For relevant parameterizations of our numerical examples, it can be verified that this assumption is satisfied. If interest rate risk and default risk are correlated, there are (pathological) cases where it is violated. More formally, note that

$$\frac{P_t^s}{\bar{P}_t^s} = \mathrm{E}_t\left[e^{-\int_t^s l_u \lambda_u \, du}\right] + \frac{\mathrm{Cov}_t[e^{-\int_t^s l_u \lambda_u \, du}, e^{-\int_t^s r_u \, du}]}{\bar{P}_t^s}.$$

As expected, the covariance between default risk and interest rate risk plays a decisive role. For instance, if $\mathrm{Cov}_t[e^{-\int_t^s l_u \lambda_u \, du}, e^{-\int_t^s r_u \, du}]$ is negative and non-increasing in s, then the ratio $P_t^s/\bar{P}_t^s$ is decreasing, but again this is only a sufficient condition. Even if the covariance increases, the ratio $P_t^s/\bar{P}_t^s$ is decreasing as long as the adjusted survival probability $\mathrm{E}_t\left[e^{-\int_t^s l_u \lambda_u \, du}\right]$ decreases faster than $\mathrm{Cov}_t[e^{-\int_t^s l_u \lambda_u \, du}, e^{-\int_t^s r_u \, du}]/\bar{P}_t^s$ increases.

Example 2 Consider the following Ho-Lee model with zero drift: $dr_t = b\, dW_t$ with a constant b, $\lambda_t = \Lambda_0 + \Lambda_1 r_t$, and a constant loss rate L. Although simple, this model is well-suited to highlight the main points, since the ratio of the zero-coupon prices can easily be calculated:

$$\frac{P_t^s}{\bar{P}_t^s} = \exp\left(b^2 L\Lambda_1(2 + L\Lambda_1)(s-t)^3 - (1 + r_t)L\Lambda_0(s-t)\right). \tag{21}$$

First note that for $\Lambda_0 > 0$ and $\Lambda_1 \in [-2/L, 0]$, the ratio (21) is always decreasing. However, if $\Lambda_1 \notin [-2/L, 0]$ and the time to maturity, $s - t$, is big enough, then this ratio starts to increase. For the parameters $\Lambda_0 = 0.01$, $\Lambda_1 = 0.005$, $b = 0.005$, and $L = 0.5$ the ratio is decreasing over the first 115 years. Figure 3 depicts the ratios for several short rate volatilities. Even for a volatility of 0.04, the ratio is decreasing over the first 16 years. Besides, if we increase Λ_1 and choose $\Lambda_1 = \Lambda_0 = 0.01$, then the ratio is decreasing over the first 80 (40) years if $b = 0.005$ ($b = 0.01$). This indicates that it is the ratio between Λ_1 and Λ_0 that is decisive, and not their absolute values. For this reason, the results are very similar for other values of the loss rate, since Λ_0 and Λ_1 are both multiplied by L.
□

[Figure 3 about here.]

As demonstrated in the Appendix Assumptions 1 and 2 lead to the following lemma, which will be used below:

Lemma 1 *Under Assumptions 1 and 2, the following inequality holds:*

$$\sum_{t_j>t} \frac{qP_t^{t_j}}{P_t^{q,t_n}} \bar{D}_t^{t_j} + \frac{P_t^{t_n}}{P_t^{q,t_n}} \bar{D}_t^{t_n} \leq \sum_{t_j>t} \frac{q\bar{P}_t^{t_j}}{\bar{P}_t^{q,t_n}} \bar{D}_t^{t_j} + \frac{\bar{P}_t^{t_n}}{\bar{P}_t^{q,t_n}} \bar{D}_t^{t_n}. \tag{22}$$

Note that the right-hand side is exactly the duration of the default-free coupon bond, but the left-hand side is generally not equal to the duration of the corresponding defaultable coupon bond.

Now we can state one of the main results of the paper:

Theorem 1 (Coupon Bond Durations) *(i) Suppose that Assumptions 1 and 2 are satisfied and that the duration of any defaultable zero-coupon bond is smaller than or equal to the duration of the equivalent default-free zero-coupon bond, i.e.* $D_t^s \leq \bar{D}_t^s$ *for any* $s \in [t, t_n]$*. Then the duration of a corporate coupon bond is smaller than the duration of the equivalent default-free bond,* $D_t^{q,t_n} \leq \bar{D}_t^{q,t_n}$*.*

(ii) Suppose that Assumption 1 is satisfied and that interest rate risk is independent of default risk and recovery risk. Then the duration of a corporate coupon bond is smaller than the duration of the equivalent default-free bond, $D_t^{q,t_n} \leq \bar{D}_t^{q,t_n}$*.*

Proof. (i) With $D_t^s \leq \bar{D}_t^s$ it is clear that the duration of the corporate coupon bond satisfies

$$D_t^{q,t_n} = \sum_{t_j>t} \frac{qP_t^{t_j}}{P_t^{q,t_n}} D_t^{t_j} + \frac{P_t^{t_n}}{P_t^{q,t_n}} D_t^{t_n} \leq \sum_{t_j>t} \frac{qP_t^{t_j}}{P_t^{q,t_n}} \bar{D}_t^{t_j} + \frac{P_t^{t_n}}{P_t^{q,t_n}} \bar{D}_t^{t_n},$$

and by Lemma 1, the right-hand side of the above inequality is less than or equal to the duration of the default-free coupon bond.

(ii) Equation (9) shows that under independence the corporate zero-coupon bond price can be written as $P_t^s = \bar{P}_t^s Q_t^s$, where $Q_t^s = \mathrm{E}[e^{-\int_t^s l_s \lambda_s \, ds}]$ is the adjusted survival probability. Since Q_t^s is certainly non-increasing in s, Assumption 2 is automatically satisfied. Moreover, Proposition 1 has shown that $\bar{D}_t^s = D_t^s$ in this case. Therefore claim (ii) follows from (i). □

Default risk has two interacting effects on the duration of coupon bonds: an effect on the duration of the zero-coupon bonds maturing at the promised payment dates of the coupon bond and an effect on the weighting of these durations of zero-coupon bonds. If the durations of corporate and Treasury zero-coupon bonds are equal, which is the case if interest rate risk is independent of default and recovery risk, then under Assumption 2 default risk reduces the duration of coupon bonds. In general, however, the durations of zero-coupon bonds are not equal and default risk can increase or decrease the duration of zero-coupon bonds. This implies that one does not average over the same numbers in (2) and (3). If $D_t^s \leq \bar{D}_t^s$, then both effects of default risk on duration go in the same direction so that the duration of a corporate coupon bond is smaller than the duration of the corresponding Treasury bond. If, however, $D_t^s > \bar{D}_t^s$, then default risk has two opposite effects on duration and the net effect is not clear. In Section 3, this was demonstrated by a numerical example.

The result in claim (ii) of the theorem is strong but comes at the price of the independence assumption. Nevertheless, it is important, since in a general setting it is demonstrated that the duration of a corporate bond can only be greater than the duration of a Treasury bond if interest rate risk and default risk risk are sufficiently correlated. The strength of the result in (i) lies in the fact that the whole problem is reduced to checking the relation between the durations of zero-coupon bonds as well as the ratio between their prices. If $D_t^s \geq \bar{D}_t^s$ for all s and the ratio $P_t^s/\bar{P}_t^s$ were increasing in s, the duration of the corporate coupon bond would be greater than that of the corresponding Treasury bond. However, we are not aware of any situation where $P_t^s/\bar{P}_t^s$ is increasing in s. Since we have provided a sufficient condition only, this does not mean that the duration of corporate coupon bonds cannot be greater than the duration of default-free coupon bonds and examples have been given in Section 3.

Next we derive an upper bound on the duration of a corporate coupon bond under the assumptions of Proposition 2, i.e. an affine default intensity and a constant loss rate. Again the proof is in the Appendix.

Proposition 4 (Upper Bound on Coupon Bond Duration) *Suppose Assumptions 1 and 2 are satisfied, the loss rate is a constant L, and the default intensity is $\lambda_t = \Lambda_0 + \Lambda_1 r_t$. Then the following conclusions hold:*

(i) The duration of a corporate coupon bond is bounded from above by a multiple of the duration of a Treasury coupon bond plus a correction term, namely

$$D_t^{q,t_n} \leq (1 + L\Lambda_1)\, \bar{D}_t^{q,t_n} + (1 + L\Lambda_1) \max_{t \leq t_j \leq t_n} C_t^{t_j}. \tag{23}$$

(ii) If the derivative of the short rate with respect to its initial value is deterministic, then

$$D_t^{q,t_n} \leq (1 + L\Lambda_1)\, \bar{D}_t^{q,t_n}. \tag{24}$$

Recall that if the default-free interest rates are described by either the Vasicek or the Ho-Lee model the correction term is zero so case (ii) applies. If $\Lambda_1 < 0$, the duration of the corporate coupon bond is then always smaller than the duration of the Treasury bond and the factor $1 + L\Lambda_1$ reduces this bound further since $1 + L\Lambda_1 < 1$. On the other hand, if $\Lambda_1 > 0$, the duration of a corporate coupon bond may be greater than the duration of a Treasury coupon bond, but it is bounded from above by the duration of a corporate coupon bond multiplied by $1 + L\Lambda_1 > 1$. Note that the duration of a corporate coupon bond need not be greater if $\Lambda_1 > 0$.

Furthermore, as shown in the example of Section 4.1, the correction term is very small in the Cox-Ingersoll-Ross model. Therefore, $(1 + \Lambda_1) \max_{t \leq s \leq t_n} C_t^s$ is negligible compared to $(1 + L\Lambda_1)\bar{D}_t^{q,t_n}$ and thus $(1 + L\Lambda_1)\bar{D}_t^{q,t_n}$ is a reliable upper bound in this model as well.

Finally, we comment on the *Macaulay duration* and its relations to our duration measure in the *Ho-Lee model*, since they are identical for Treasury bonds, but different for corporate bonds. In our framework, one obtains the Macaulay durations of Treasury and corporate coupon bonds by formally setting $\bar{D}_t^s = D_t^s = s - t$. By Theorem 1, the Macaulay duration of a corporate coupon bond is thus always smaller than the Macaulay duration of a Treasury coupon bond. On the other hand, the duration of a Treasury zero-coupon bond in the Ho-Lee model equals $\bar{D}_t^s = s - t$, i.e. it is identical to the Macaulay duration. However, under the assumptions of Proposition 3, the duration of a corporate zero-coupon bond is given by $(1 + L\Lambda_1)(T - t)$ and can thus be greater or smaller than the duration of a corporate zero-coupon bond. By Proposition 4, the same is true for coupon bonds.

5 Practical Implications for Risk Management

A manager of a corporate bond portfolio is interested in answers to the following questions: (1) When is the error of disregarding default risk positive or negative? (2) In which case is it more relevant to not disregard default risk?

If interest rate risk is negatively correlated with default risk the answer to the first question is straightforward: the duration of a default-free coupon bond is an upper bound for the duration of a corporate coupon bond (Theorem 1). Moreover, Proposition 4 provides an even sharper bound. However, care needs to be taken if interest rate risk is positively correlated with default risk. In this case, the duration of a corporate zero-coupon bond is greater than the duration of a Treasury zero-coupon bond (Propositions 2 and 3), but for coupon bonds the results are mixed. This is because default risk now has two opposite effects on duration (see the discussion after Theorem 1). For smaller maturities, coupon bonds behave similarly to zero-coupon bonds and thus the duration of corporate bonds is greater than the duration of Treasury bonds. If maturity increases, this changes and eventually the corporate duration falls below the Treasury duration. Figure 4 shows this behavior in the Cox-Ingersoll-Ross model, where $dr_t = \kappa(\theta - r_t)\,dt + \sigma\sqrt{r_t}\,dW_t$. For $r_0 = 0.03$, $\kappa = 0.1$, $\theta = 0.078$, $\sigma = 0.05$, $\Lambda_0 = 0.01$, and $L = 0.5$, the thick upper line depicts the difference $D_t^q - \bar{D}_t^q$ between the Treasury and corporate duration for bonds with annual coupons of 5% and $\Lambda_1 = 0.01$, i.e. if interest rate and default risk are positively correlated. The thick lower line depicts the same difference for $\Lambda_1 = -0.01$ implying negative correlation. The differences in using either the correct or the approximated upper bound for D_t^{q,t_n} in (23) and (24) instead of the true value of D_t^{q,t_n} are plotted above the thick lines. For $\Lambda_1 = 0.01$ the corporate duration is greater up to a maturity of 10 years and then it becomes smaller, whereas for $\Lambda_1 = -0.01$ the corporate duration is always smaller. In both cases, the upper bounds are very similar, which could be expected from our above example showing that the correction term is very small. Besides, the upper bounds are tight for the first three years.

The answer to the second question is also mixed: Since the difference between the durations for $\Lambda_1 = 0.01$ changes its sign, the absolute value is smaller than for $\Lambda_1 = -0.01$. For this reason, disregarding default risk is more problematic if it is negatively correlated with interest rate risk.

[Figure 4 about here.]

6 Conclusion

The literature on the duration of corporate bonds is very sparse despite the obvious applications of (appropriately defined) duration in interest rate risk management. Most existing papers conclude that the duration of a corporate bond is smaller than the duration of a similar default-free bond. In this paper we have offered a much more detailed comparison of the durations of defaultable and default-free bonds and we have arrived at a more balanced conclusion.

The relation between the default intensity and the default-free interest rate is crucial for the duration of the corporate bond. If the default intensity is increasing (decreasing) in the default-free short rate, the duration of a corporate zero-coupon bond is greater (less) than that of an equivalent default-free bond, at least in affine settings. The duration of a (corporate or default-free) coupon bond is a weighted average of the durations of the embedded (corporate or default-free, respectively) zero-coupon bonds. Under weak assumptions, the weights applied to the short-maturity, low-duration zero-coupon bonds are higher for corporate coupon bonds than for default-free bonds. Hence, if the corporate zero-coupon bonds have lower (or slightly greater) durations than their default-free counterparts, a corporate coupon bond will have a lower duration that of a default-free coupon bond. However, empirical studies indicate that, at least for some corporate bonds, the default intensity is increasing so fast in the default-free interest rate that the duration of the corporate coupon bond may very well exceed that of the equivalent default-free bond.

A Proofs

Proof of Proposition 2.

(i) The derivative (13) can be rewritten as

$$
\begin{aligned}
-\frac{\partial P_t^s}{\partial r} &= (1+L\Lambda_1)\mathrm{E}_t\left[e^{-\int_t^s (r_u+L\lambda_u)\,du}\int_t^s y_u\,du\right] \\
&= (1+L\Lambda_1)\mathrm{E}_t\left[e^{-\int_t^s r_u\,du}\int_t^s y_u\,du\right]\mathrm{E}_t\left[e^{-\int_t^s L\lambda_u\,du}\right] \\
&\quad + (1+L\Lambda_1)\mathrm{Cov}_t\left[e^{-\int_t^s r_u\,du}\int_t^s y_u\,du\,,\,e^{-\int_t^s L\lambda_u\,du}\right] \\
&= -\frac{\partial \bar{P}_t^s}{\partial r}(1+L\Lambda_1)Q_t^s + (1+L\Lambda_1)\mathrm{Cov}_t\left[e^{-\int_t^s r_u\,du}\int_t^s y_u\,du\,,\,e^{-\int_t^s L\lambda_u\,du}\right],
\end{aligned}
$$

where $Q_t^s = \mathrm{E}_t\left[e^{-\int_t^s L\lambda_u\,du}\right]$ denotes the firm's (recovery adjusted) survival probability. Therefore, the duration of the bond is given by

$$
D_t^s = (1+L\Lambda_1)\frac{\bar{P}_t^s Q_t^s}{P_t^s}\bar{D}_t^s + \frac{1+L\Lambda_1}{P_t^s}\mathrm{Cov}_t\left[e^{-\int_t^s r_u\,du}\int_t^s y_u\,du, e^{-\int_t^s L\lambda_u\,du}\right].
$$

Since $\bar{P}_t^s Q_t^s = P_t^s - \mathrm{Cov}_t[e^{-\int_t^s r_u\,du}, e^{-\int_t^s L\lambda_u\,du}]$, the result follows.

(ii) In this case, $\int_t^s y_u\,du = \int_t^s \mathrm{E}_t^{Q^s}[y_u]\,du = \bar{D}_t^s$. Substituting this relation into C_t^s gives the desired result. □

Proof of Proposition 3.

We wish to calculate

$$
P_t^s = \mathrm{E}\left[e^{-\int_t^s (r_u+L\lambda_u)\,du}\right] = e^{-L\Lambda_0(s-t)}\mathrm{E}\left[e^{-\int_t^s \tilde{r}_u\,du}\right],
$$

where $\tilde{r} = k_1 r$ and $k_1 = 1 + L\Lambda_1$. By Ito's lemma, the auxiliary process $\tilde{r}$ satisfies

$$
d\tilde{r}_t = (k_1K_0 + K_1\tilde{r}_t)dt + \sqrt{k_1^2H_0 + \tilde{H}_1\tilde{r}_t}\,dW_t,
$$

where $\tilde{H}_1 = k_1H_1$. By Duffie, Pan, and Singleton (2000), we obtain $\mathrm{E}\left[e^{-\int_t^s \tilde{r}_u\,du}\right] = e^{-\tilde{A}(s-t)-B(s-t)\tilde{r}_t}$, where $\tilde{A}$ is a deterministic function and B satisfies (20) if H_1 is replaced by $\tilde{H}_1$. Since the duration can be rewritten as $D_t^s = (1+L\Lambda_1)\bar{D}_t^s + (1+L\Lambda_1)\{B(s-t) - \bar{B}(s-t)\}$, the result for the correction term follows. Finally, note that $A(\tau) = L\Lambda_0\tau + \tilde{A}(\tau)$, where $\tilde{A}(\tau) = \mathcal{A}(\tau; k_1K_0, K_1, k_1^2H_0, k_1H_1)$. □

Proof of Lemma 1.

Any non-decreasing bounded, measurable real-valued function on $[t, t_n]$ can be approximated by a sum of indicator functions $\mathbf{1}_{[t', t_n]}$ with $t' \in [t, t_n]$. Hence, it is sufficient to prove the claim for $\bar{D}_t^s = \mathbf{1}_{[t',t_n]}(s)$, $s \in [t, t_n]$. In this case the difference between the left-hand side and the right-hand side of (22) is

$$\begin{aligned}\sum_{t_j>t}\left(\frac{qP_t^{t_j}}{P_t^{q,t_n}}-\frac{q\bar{P}_t^{t_j}}{\bar{P}_t^{q,t_n}}\right)\bar{D}_t^{t_j}+\left(\frac{P_t^{t_n}}{P_t^{q,t_n}}-\frac{\bar{P}_t^{t_n}}{\bar{P}_t^{q,t_n}}\right)\bar{D}_t^{t_n}\\ =\sum_{t_j>t'}\left(\frac{qP_t^{t_j}}{P_t^{q,t_n}}-\frac{q\bar{P}_t^{t_j}}{\bar{P}_t^{q,t_n}}\right)+\left(\frac{P_t^{t_n}}{P_t^{q,t_n}}-\frac{\bar{P}_t^{t_n}}{\bar{P}_t^{q,t_n}}\right).\end{aligned}\tag{25}$$

We want to show that this difference is non-positive. Since $P_t^s/\bar{P}_t^s$ is non-increasing in s and both the terms $\frac{qP_t^{t_j}}{P_t^{q,t_n}}$, $t_j\in[t,t_n]$, and the terms $\frac{q\bar{P}_t^{t_j}}{\bar{P}_t^{q,t_n}}$, $t_j\in[t,t_n]$, sum up to one, we can find $t^*\in[t,t_n]$ so that $\frac{qP_t^{t_j}}{P_t^{q,t_n}}\geq\frac{q\bar{P}_t^{t_j}}{\bar{P}_t^{q,t_n}}$ for $t_j\in[t,t^*]$.

Case 1, $t^*\leq t'$: Here $\frac{qP_t^{t_j}}{P_t^{q,t_n}}\leq\frac{q\bar{P}_t^{t_j}}{\bar{P}_t^{q,t_n}}$ for all $t_j>t'$, so that all the terms in the sum in (25) are non-positive and the result follows.

Case 2, $t^*>t'$: In this case,

$$\begin{aligned}&\sum_{t_j>t'}\left(\frac{qP_t^{t_j}}{P_t^{q,t_n}}-\frac{q\bar{P}_t^{t_j}}{\bar{P}_t^{q,t_n}}\right)+\left(\frac{P_t^{t_n}}{P_t^{q,t_n}}-\frac{\bar{P}_t^{t_n}}{\bar{P}_t^{q,t_n}}\right)\\ &\quad\leq\sum_{t_j\leq t'}\left(\frac{qP_t^{t_j}}{P_t^{q,t_n}}-\frac{q\bar{P}_t^{t_j}}{\bar{P}_t^{q,t_n}}\right)+\sum_{t_j>t'}\left(\frac{qP_t^{t_j}}{P_t^{q,t_n}}-\frac{q\bar{P}_t^{t_j}}{\bar{P}_t^{q,t_n}}\right)+\left(\frac{P_t^{t_n}}{P_t^{q,t_n}}-\frac{\bar{P}_t^{t_n}}{\bar{P}_t^{q,t_n}}\right)\\ &\quad=\sum_{t_j>t}\left(\frac{qP_t^{t_j}}{P_t^{q,t_n}}-\frac{q\bar{P}_t^{t_j}}{\bar{P}_t^{q,t_n}}\right)+\left(\frac{P_t^{t_n}}{P_t^{q,t_n}}-\frac{\bar{P}_t^{t_n}}{\bar{P}_t^{q,t_n}}\right)\\ &\quad=\left(\sum_{t_j>t}\frac{qP_t^{t_j}}{P_t^{q,t_n}}+\frac{P_t^{t_n}}{P_t^{q,t_n}}\right)-\left(\sum_{t_j>t}\frac{q\bar{P}_t^{t_j}}{\bar{P}_t^{q,t_n}}+\frac{\bar{P}_t^{t_n}}{\bar{P}_t^{q,t_n}}\right)\\ &\quad=1-1=0,\end{aligned}$$

which shows the result. □

Proof of Proposition 4.

Since $D_t^s=(1+L\Lambda_1)\{\bar{D}_t^s+C_t^s\}$, we obtain

$$\begin{aligned}D_t^{q,t_n}&=\sum_{t_j>t}\frac{qP_t^{t_j}}{P_t^{q,t_n}}(1+L\Lambda_1)(\bar{D}_t^{t_j}+C_t^{t_j})+\frac{P_t^{t_n}}{P_t^{q,t_n}}(1+L\Lambda_1)(\bar{D}_t^{t_n}+C_t^{t_n})\\ &\leq(1+L\Lambda_1)\left(\sum_{t_j>t}\frac{qP_t^{t_j}}{P_t^{q,t_n}}\bar{D}_t^{t_j}+\frac{P_t^{t_n}}{P_t^{q,t_n}}\bar{D}_t^{t_n}\right)+(1+L\Lambda_1)\max_{t\leq t_j\leq t_n}C_t^{t_j}\\ &\leq(1+L\Lambda_1)\bar{D}_t^{q,t_n}+(1+L\Lambda_1)\max_{t\leq t_j\leq t_n}C_t^{t_j},\end{aligned}$$

where the first inequality is due to the maximum and the second is due to Lemma 1. Claim (ii) is valid since in this case $C_t^{t_j}=0$. □

References

Acharya, V. V. and J. N. Carpenter, 2002. Corporate Bond Valuation and Hedging with Stochastic Interest Rates and Endogenous Bankruptcy. Review of Financial Studies 15(5), 1355–1383.

Babbel, D. F., C. Merrill, and W. Panning, 1997. Default Risk and the Effective Duration of Bonds. Financial Analysts Journal 53(1), 35–44.

Bakshi, G., D. Madan, and F. Zhang, 2006. Investigating the Role of Systematic and Firm-Specific Factors in Default Risk: Lessons from Empirically Evaluating Credit Risk Models. Journal of Business 79(4), 1955–1988.

Chance, D. M., 1990. Default Risk and the Duration of Zero Coupon Bonds. Journal of Finance 45(1), 265–274.

Cox, J. C., J. E. Ingersoll, Jr., and S. A. Ross, 1979. Duration and the Measurement of Basis Risk. Journal of Business 52(1), 51–61.

Cox, J. C., J. E. Ingersoll, Jr., and S. A. Ross, 1985. A Theory of the Term Structure of Interest Rates. Econometrica 53(2), 385–407.

Duffee, G. R., 1998. The Relation between Treasury Yields and Corporate Bond Yield Spreads. Journal of Finance 53(6), 2225–2241.

Duffie, D., J. Pan, and K. Singleton, 2000. Transform Analysis and Asset Pricing for Affine Jump-Diffusions. Econometrica 68(6), 1343–1376.

Duffie, D. and K. Singleton, 1999. Modeling Term Structures of Defaultable Bonds. Review of Financial Studies 12(4), 687–720.

Fooladi, I. J., G. S. Roberts, and F. Skinner, 1997. Duration for Bonds with Default Risk. Journal of Banking and Finance 21, 1–16.

Ingersoll, Jr., J. E., J. Skelton, and R. Weil, 1978. Duration Forty Years Later. Journal of Financial and Quantitative Analysis 13(4), 627–650.

Jacoby, G., 2003. A Duration Model for Defaultable Bonds. Journal of Financial Research 26, 129–146.

Jarrow, R. A. and Y. Yildirim, 2002. Valuing Default Swaps Under Market and Credit Risk Correlation. Journal of Fixed Income 11(4), 7–19.

Longstaff, F. A. and E. S. Schwartz, 1995. A Simple Approach to Valuing Risky Fixed and Floating Rate Debt. Journal of Finance 50(3), 789–819.

Merton, R. C., 1974. On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. Journal of Finance 29, 449–470.

Papageorgiou, N. and F. S. Skinner, 2006. Credit Spreads and the Zero-Coupon Treasury Spot Curve. Journal of Financial Research 29(3), 421–439.

Protter, P., 2005. Stochastic Integration and Differential Equations (Second ed.), Volume 21 of Applications of Mathematics. Springer-Verlag.

Vasicek, O., 1977. An Equilibrium Characterization of the Term Structure. Journal of Financial Economics 5, 177–188.

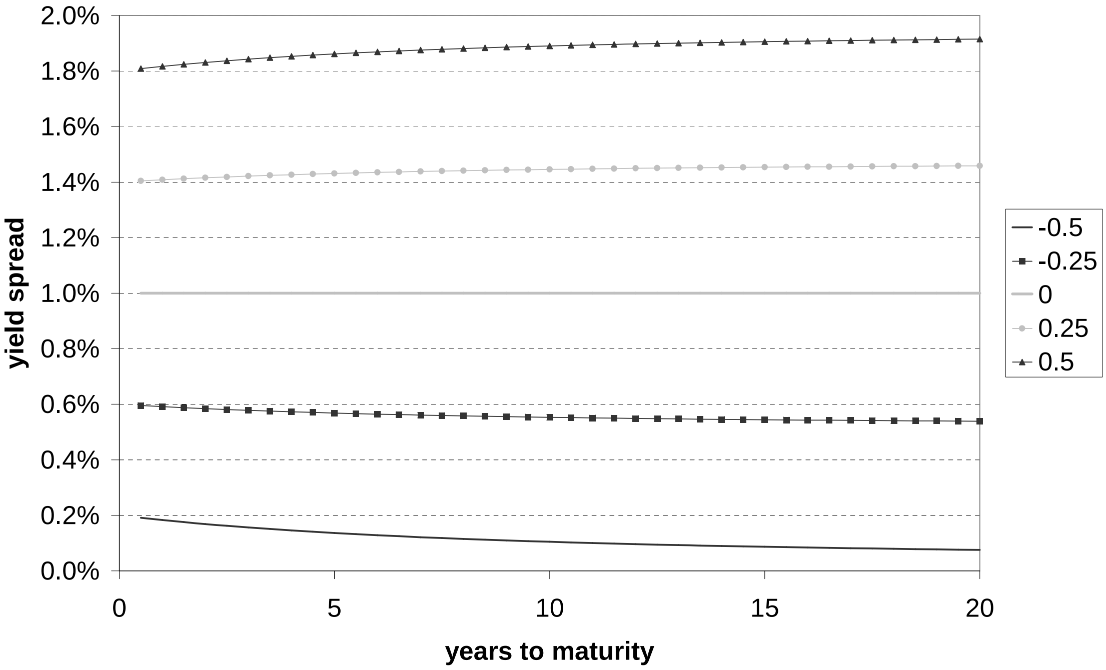

Figure 1: **Zero-coupon yield spreads.** The figure shows the difference between the yield of a corporate zero-coupon bond and a default-free zero-coupon bond as a function of time to maturity. The different curves are for different values of the parameter Λ_1 ranging from -0.5 (lower curve) to 0.5 (upper curve). The current default-free short rate is 4%. The parameters of the short rate process are $\kappa = 0.15$, $\sigma_r = 0.01$, and $\theta = 0.0522$ so that the asymptotic zero-coupon yield is 5%. The loss in case of default is $L = 0.4$ and the fixed part of the default intensity is $\Lambda_0 = 0.025$.

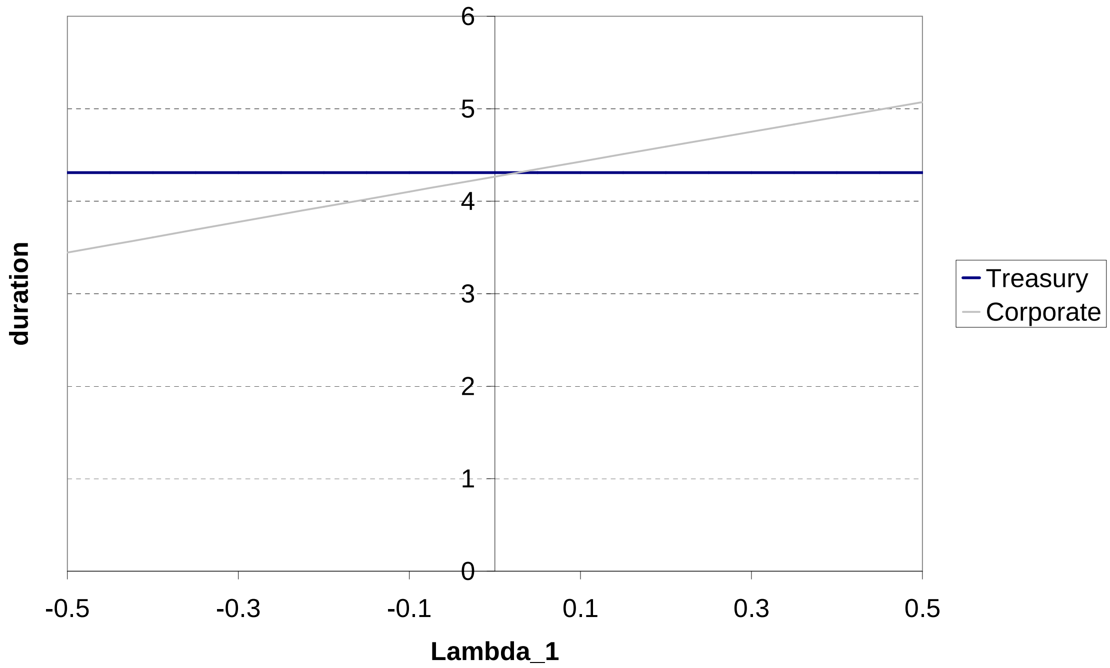

Figure 2: **The durations of 10-year bonds.** The figure shows the duration of a 10-year corporate coupon bond as a function of the interest rate sensitivity of the default intensity rate (the parameter Λ_1). The horizontal line shows the duration of the similar Treasury bond. The coupon is 6% per year with semi-annual payments. The current default-free short rate is 4%. The parameters of the short rate process are $\kappa = 0.15$, $\sigma_r = 0.01$, and $\theta = 0.0522$ so that the asymptotic zero-coupon yield is 5%. The loss in case of default is $L = 0.4$ and the fixed part of the default intensity is $\Lambda_0 = 0.025$.

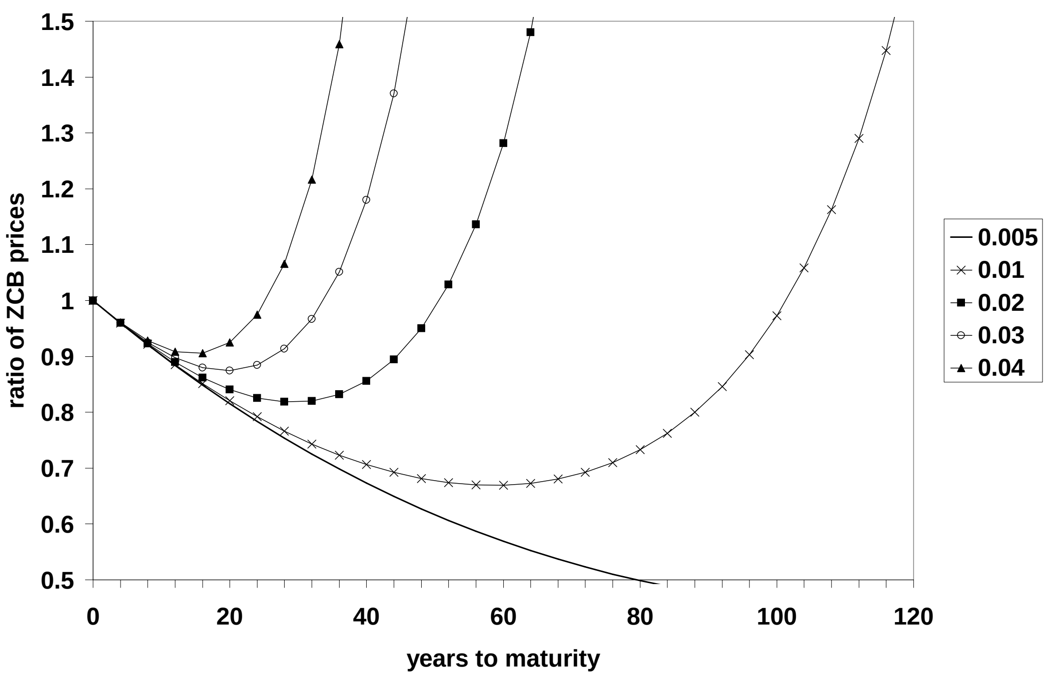

Figure 3: **The ratio of zero-coupon bond prices.** The figure shows the ratio of the price of a defaultable zero-coupon bond to the price of a default-free zero-coupon bond of the same maturity as a function of the time to maturity. The prices are computed using the Ho-Lee model $dr_t = b\,dW_t$, a constant loss rate of $L = 0.5$, and a default intensity of $\lambda_t = 0.01 + 0.005\,r_t$. Each curve corresponds to a given value of the short rate volatility b.

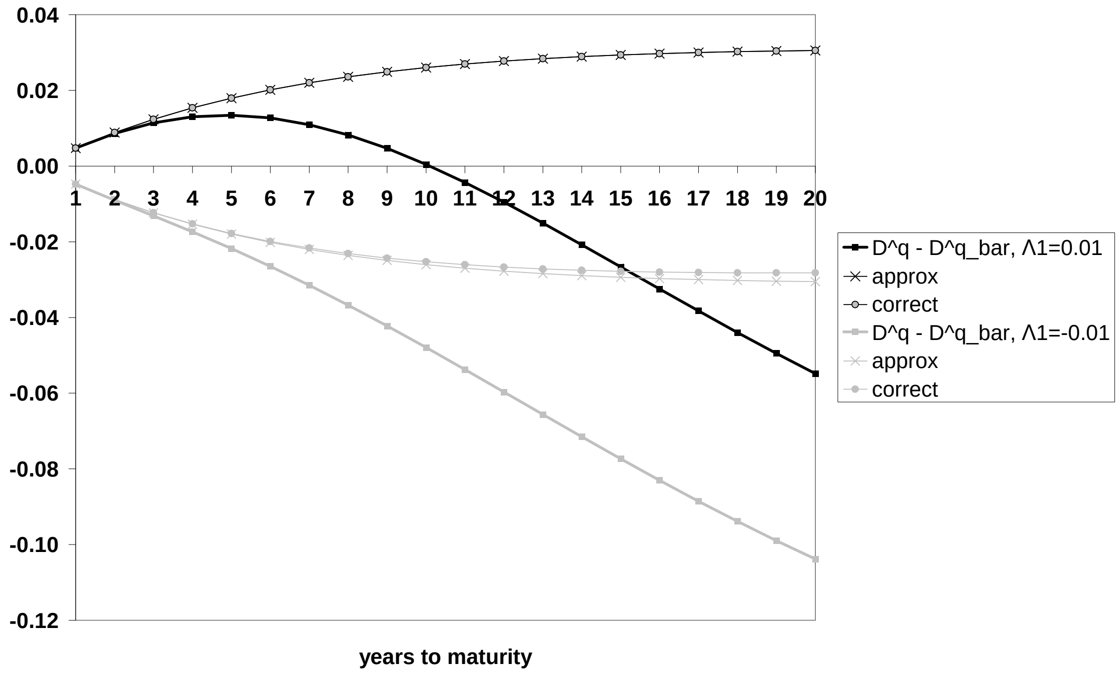

Figure 4: **Coupon bond durations in the CIR model.** The figure shows the difference between the durations of the defaultable and the default-free coupon bonds as a function of time to maturity. The values are computed using the Cox-Ingersoll-Ross model $dr_t = \kappa(\theta - r_t)\, dt + \sigma\sqrt{r_t}\, dW_t$ with $r_0 = 0.03$, $\kappa = 0.1$, $\theta = 0.078$, and $\sigma = 0.05$. The loss rate is assumed constant and equal to $L = 0.5$, and the default intensity is affine, $\lambda_t = \Lambda_0 + \Lambda_1 r_t$, with $\Lambda_0 = 0.01$. The black curves are for $\Lambda_1 = 0.01$ and the grey curves for $\Lambda_1 = -0.01$. In each case the thick line with the boxes shows the exact difference in durations, $D_t^{q,t_n} - \bar{D}_t^{q,t_n}$, while the two other curves show the duration difference when D_t^{q,t_n} is replaced by either the correct upper bound defined in (23) or the approximate upper bound defined in (24). For the case of $\Lambda_1 = 0.01$ the two curves involving the upper bounds are virtually indistinguishable.

Parameter	Value	Interpretation
r_0	0.04	initial short rate
κ	0.15	mean reversion speed
σ_r	0.01	short rate volatility
θ	0.0522	risk-neutral average short rate
q	0.03	coupon rate (not annualized)
$t_{j+1} - t_j$	0.5 years	coupon frequency
$t_n - t_0$	10 years	bond maturity
L	0.4	loss rate given default
Λ_0	0.025	constant part of default intensity
Λ_1	in $[-0.5, 0.5]$	interest rate sensitivity of default intensity

Table 1: **Parameters in the affine example.** The table lists the values of the affine model parameters used in the computation of the durations of corporate and Treasury coupon bonds.